Effective inseparability and c.e. structures

Andrea Sorbi
(Joint work with U. Andrews)

Department of Information Engineering and Mathematics
University of Siena
53100 Siena, Italy

October, 2024

Computability and Mathematical Definability
70-th of Ted

UNIVERSITÀ DI SIENA 1240

My papers with Ted

- (1998 Annali di Matematica Pura ed Applicata, Quasi-minimal enumeration degrees and minimal Turing degrees) There exists a set A of quasiminimal e-degree which is e-below uncountably many sets B of minimal Turing degree.
- (2005 J. Math. Logic, On extensions of embeddings into the enumeration degrees of the Σ_2^0 sets, with Steffen Lempp). The paper gives an algorithm for deciding whether an embedding of a finite partial order P into the enumeration degrees of the Σ_2-sets can always be extended to an embedding of a finite partial order $Q \supset P$.
- (2014 J. Symbolic Logic, A note on initial segments of the enumeration degrees) There is no linearly ordered nontrivial initial segment of the enumeration degrees.

Definition [Kleene, Smullyan] A disjoint pair (A, B) of c.e. sets of natural numbers is effectively inseparable (or, simply, e.i.) if there exists a partial computable function $\psi(u, v)$ (called a productive function for the pair) such that

$$(\forall u, v)[A \subseteq W_u \,\&\, B \subseteq W_v \,\&\, W_u \cap W_v = \emptyset \Rightarrow \psi(u, v)\downarrow \,\&\, \psi(u, v) \notin W_u \cup W_v].$$

R. Smullyan. Theory of Formal Systems. Princeton University Press, Princeton, New Jersey, 1961. Annals of Mathematical Studies Vol 47

Let T be any consistent c.e. extension of Robinson's system R, or Q. Then

- the pair $(\mathrm{Thm}_T, \mathrm{Ref}_T)$ is e.i. (T is said to be an e.i. theory), where:

$$\mathrm{Thm}_T = \{\alpha \in \mathrm{Sent} : T \vdash \alpha\}$$
$$\mathrm{Ref}_T = \{\alpha \in \mathrm{Sent} : T \vdash \neg\alpha\}$$

- T is essentially undecidable.

Proof If $(A, B), (C, D)$ are pairs of disjoint c.e. sets, (A, B) is e.i., and $(A, B) \subseteq (C, D)$ then (C, D) is e.i.. □

By a c.e structure, or a positive structure, A we will mean a nontrivial algebraic-relational structure for which there exists a positive presentation, i.e. a structure A_ω of the same type as A but with universe ω and possessing uniformly computable operations, uniformly c.e. relations, and a c.e. equivalence relation $=_A$ which is a congruence on A_ω such that A is isomorphic with A_ω divided by $=_A$ (i.e. $A \simeq A_{\omega/=_A}$).

Example The Lindenbaum Boolean algebra L_T of the sentences of a c.e. consistent theory T is (modulo Gödel coding) a positive Boolean algebra.

A c.e. equivalence relation will be called a ceer.

See [Selivanov 2003] for a masterly introduction to c.e. structures.

V. Selivanov. Positive structures. In S.B. Cooper and S.S. Goncharov, editors, Computability and Models, pages 321–350. Springer, New York, 2003

Definition A positive Boolean algebra B is effectively inseparable if the pair of $=_B$-equivalence classes $(0_B, 1_B)$ is e.i. (we write for simplicity $x_B = [x]_{=_B}$, and $0, 1$ are supposed to present the least and the greatest element of B). (Warning: "e.i." will tacitly assume "positive".)

Example The Lindenbaum algebra L_{PA} of PA is an e.i. Boolean algebra.

Theorem [Computable Isomorphism Thm for e.i. BAs [Pour-El and S. Kripke, 1967]]
All e.i. Boolean algebras are computably isomorphic.

Effectively inseparability and Boolean algebras, II

Theorem [Pour El and Kripke, Montagna and S. 1985] If B is an e.i. Boolean algebra then the following universality properties hold: B computably embeds

- all positive Boolean algebras;
- all positive bounded distributive lattices (and all positive distributive lattices);
- all positive posets.

By computable embeddings and computable isomorphisms we mean of course embeddings and isomorphisms, respectively, "induced" by computable functions.

Corollary Every e.i. Boolean algebra is locally universal, i.e. the above universality results (with respect to the classes of positive structures mentioned in the theorem) hold in every nontrivial interval of the algebra.

Proof Obvious since every nontrivial interval is an e.i. Boolean algebra as well.

Effectively inseparability and Boolean algebras, III

Definition If $\preceq$ is a preordering relation on ω (with $\equiv$ its associated equivalence relation), then we say that $\preceq$ is uniformly dense, if there exists a computable function f such that for every a, b if $a \prec b$ then

- $a \prec f(a, b) \prec b$,
- if $a \equiv a'$ and $b \equiv b'$ then $f(a, b) \equiv f(a', b')$.

Theorem [Uniform Density Theorem for e.i. Boolean algebras [Shavrukov and Visser 2014] Every e.i. Boolean algebra B is uniformly dense, i.e. the preordering $\leq_B$ is uniformly dense.

M. B. Pour-El and S. Kripke. Deduction preserving "Recursive Isomorphisms" between theories. Fund. Math., 61:141–163, 1967

F. Montagna and A. Sorbi. Universal recursion theoretic properties of r.e. preordered structures. J. Symbolic Logic, 50(2):397–406, 1985

V. Yu. Shavrukov and A. Visser. Uniform density in Lindenbaum algebras. Notre Dame J. Form. Log., 55(4):569–582, 2014

Definition A positive bounded lattice L is effectively inseparable if the pair of $=_L$-equivalence classes $(0_L, 1_L)$ is e.i.. (Again: "e.i." assumes "positive".)

Example The Lindenbaum algebra L_{HA} of Heyting Arithmetic is an e.i. lattice.

In fact, the Gödel-Gentzen double-negation translation ensures that there is a computable mapping, which 1-reduces the disjoint pairs $(0_{PA}, 1_{PA}) \leq_1 (0_{HA}, 1_{HA})$, implying that $(0_{HA}, 1_{HA})$ is e.i. as so is $(0_{PA}, 1_{PA})$.

What does it survive when we go from e.i. Boolean algebras to e.i. lattices?

Of course we cannot expect that all the previous results for Boolean algebras extend to lattices or even distributive lattices. For instance:

- We cannot expect a computable isomorphism theorem: lattices can be distributive, non-distributive, Heyting algebras, Boolean algebras, etc.
- "universality" fails. For instance:

  Theorem [Andrews and S., 2021] E.i. distributive lattices are not necessarily universal with respect to the class of positive distributive lattices.

U. Andrews and A. Sorbi. Effective inseparability, lattices, and pre-ordering relations. Review of Symbolic Logic, 14 (4) pages 838-865, 2021

However, define:

Definition A positive preordering $\preceq$ is universal, if for every positive preordering R there exists a computable function f such that

$$(\forall x, y\,[x\,R\,y \Leftrightarrow f(x) \preceq f(y)].$$

$\preceq$ is is locally universal if every interval $[a, b]_{\preceq}$ wth $a \prec b$ computably embeds every positive preorder.
(In other words, from now on, universal means universal with respect to the class of positive preorders.)

Then in the rest of the talk we will see:

- An e.i. lattice L is universal and locally universal (hence, in going from e.i. Boolean algebras to e.i. lattices, universality and local universality are preserved only with respect to positive preorders).
- An e.i. lattice is uniformly dense i.e. its preordering relation is uniformly dense.

[Basic Lemma [Andrews and S.]] Let A be a positive algebra whose type contains two binary operations $+, \cdot$, and two constants (presented by the numbers) $0, 1$ such that $+$ is associative, the pair of sets $(0_A, 1_A)$ is e.i. and, for every a,

$$0 + a =_A a + 0 =_A a, \qquad a \cdot 0 =_A 0, \qquad a \cdot 1 =_A a.$$

Then $=_A$ is a uniformly finitely precomplete (u.f.p.) ceer.

Corollary If L is an e.i. lattice then the ceer $=_L$ is u.f.p. .

Definition [Montagna 1982, Shavrukov 1996] A nontrivial equivalence relation E on ω is uniformly finitely precomplete (u.f.p.) if it has a u.f.p. totalizer, i.e. a (total) computable function $f(D, e, x)$ such that

$$(\forall D, e, x)\, [\varphi_e(x)\downarrow \,\&\, \varphi_e(x) \in [D]_E \Rightarrow \varphi_e(x)\ E\ f(D, e, x)]\,.$$

F. Montagna. Relative precomplete numerations and arithmetic. J. Philosphical Logic, 11(4):419–430, 1982

V. Yu. Shavrukov. Remarks on uniformly finitely precomplete positive equivalences. Math. Log. Quart., 42:67–82, 1996

Proof of the Basic Lemma: Need to find a totalizer $f(D,e,x)$ such that $\varphi_e(x)\downarrow\in [D]_{=_A} \Rightarrow f(D,e,x) =_A \varphi_e(x)$

Let p be a (total) productive function for $(0_A, 1_A)$. Given D, e, x and $d \in D$ (here the numbers $u_{d,D,e,x}$ and $v_{d,D,e,x}$ form a computable set of fixed points we control by the Recursion Theorem: for simplicity write $u = u_{d,D,e,x}$ and $v = v_{d,D,e,x}$) let:

$$W_u = \begin{cases} 0_A \cup \{p(u,v)\}, & \text{if we first see } \varphi_e(x) =_A d, \\ 0_A, & \text{otherwise} \end{cases}$$

$$W_v = \begin{cases} 1_A \cup \{p(u,v)\}, & \text{if we first see } \varphi_e(x) =_A d' \in D,\ d' \neq d, \\ 1_A & \text{otherwise} \end{cases}$$

So:

- if $\varphi_e(x) =_A d$ first, then $p(u,v) =_A 1$,
- if $\varphi_e(x) =_A d'$ first to a $d' \in D$, $d' \neq d$, then $p(u,v) =_A 0$.

It follows that $f(D,e,x) = \sum_{d' \in D} d' \cdot p(u_{d',D,e,x}, v_{d',D,e,x})$ is a suitable u.f.p. totalizer.

Theorem [Universality Theorem [Andrews and S.]]
If L is an e.i. lattice then L is universal.

Remark The lattice structure is needed! There exists an e.i. bounded upper semi-lattice U such that $\leq_U$ is not universal, [Andrews and S.].

However, distributivity is not needed!

Corollary Every e.i. lattice is locally universal.

Proof Let L be an e.i. lattice and let $a <_L b$ in L. It is easy to see that every disjoint pair of equivalence classes of a u.f.p. ceer is (uniformly) e.i. .
Therefore, the pair (a_L, b_L) is e.i., and so the interval $[a, b]_{\leq_L}$ is an e.i. bounded lattice, and therefore universal, by the theorem.

A preliminary observation:

Claim If L is e.i., then there is a computable function $k(a, b, D, e, x)$ (called a local u.f.p. totalizer) so that

$$a \leq_L b \Rightarrow k(a, b, D, e, x) \in [a, b]_{\leq_L}$$

and if $\varphi_e(x) =_L d \in D$ for some $d \in D$ then

$$k(a, b, D, e, x) =_L (d \wedge b) \vee a.$$

In particular, if $\varphi_e(x) =_L d \in D$ for some $d \in D$ with $d \in [a, b]_{\leq_L}$, then $k(a, b, D, e, x) =_L d$.

Proof Let $k(a, b, D, e, x) = (j(D, e, x) \wedge b) \vee a$, where $j(D, e, x)$ is a u.f.p. totalizer of $=_L$.

Let R be a given positive preorder. We need a computable function f such that

$$m \, R \, n \Leftrightarrow f(m) \leq_L f(n).$$

How to define $f(n)$, supposing we have $f(j)$, for all $j < n$?
Let us denote by σ any generic pre-ordering configuration that one can have on the interval $[0, n]$ of natural numbers, with λ denoting the antichain (which we may assume is also the configuration of R at stage 0). Let X_σ be the set of elements strictly σ-below n, and let Y_σ be the set of elements which are strictly σ-above n.
Partially order these configurations by $\tau \preceq \sigma$ if $X_\tau \subseteq X_\sigma$ and $Y_\tau \subseteq Y_\sigma$. Denote $a_\sigma = \bigvee f(X_\sigma)$ and $b_\sigma = \bigwedge f(Y_\sigma)$.
For each such σ we have a dedicated fixed point e_σ which we control by the Recursion Theorem.

Sketch of proof of the Universality Theorem, III

Let $k(a, b, D, e, x)$ be a local u.f.p. totalizer for $=_L$, and define $x_\nu = k(a_\nu, b_\nu, \{0, 1\}, e_\nu, 0)$ for the leaf ν, corresponding to the total preordering on $[0, n]$).

Finally, for every σ define

$$x_\sigma = k(a_\sigma, b_\sigma, \{0, 1, x_\tau : \sigma \prec \tau\}, e_\sigma, 0)$$

and

$$f(n) = x_\lambda.$$

We now specify how to compute $\varphi_{e_\sigma}(0)$ (by stages, starting from $\varphi_{e_\sigma}(0)\uparrow$).

At stage $s + 1$, if the preordering configuration of R at s was σ, and at $s + 1$ it has evolved to τ (with clearly $\sigma \prec \tau$), then define (as $\varphi_{e_\sigma}(0)$ is still undefined)

$$\varphi_{e_\sigma}(0) = x_\tau.$$

So, $x_\sigma =_L x_\tau$. Notice that this leaves $\varphi_{e_\tau}(0)$ still undefined.

By properties of the local u.f.p. totalizer, and by induction, it is easy to see that $x_\lambda =_L x_\tau$ and $f(n) = x_\lambda =_L x_\tau \in [a_\tau, b_\tau]_{\leq_L}$.

Eventually we stop acting on $f(n)$, and we stop when we make a definition $\varphi_{e_\sigma}(0) = x_\rho$ where ρ is the final preordering configuration of R on $[0, n]$.

Thus we get, for all $m < n$,

$$m \; R \; n \Rightarrow f(m) \leq_L f(n) \quad \text{and } n \; R \; m \Rightarrow f(n) \leq_L f(m).$$

On the other hand at no stage s does the construction show that $f(m) \leq_L f(n)$, but $m \not R n$. Should we see this, we would be able at that stage to take advantage of the possibility of making more definitions for some of those $\varphi_{e_\rho}(0)$ which are still undefined to force $f(m) =_L 1$ and $f(n) =_L 0$, thus getting $0 =_L 1$, "killing" the construction. (Notice that $\varphi_{e_\nu}(0)$ is certainly undefined, unless we have defined it already to kill already the construction.) □

Second application of e.i.-ness: uniform density

Theorem [Uniform Density Theorem for e.i. lattices [Andrews and S.]]
If L is an e.i. lattice then L is uniformly dense, i.e. the associated pre-ordering relation $\leq_L$ is uniformly dense.

Sketch of proof of the Uniform Density Theorem, I

By the Basic Lemma we have a computable function $k(D, e, x)$ witnessing that $=_L$ is u.f.p. Let $\{e_{a,b} : a, b \in \omega\}$ be a computable list of indices we control by the Recursion Theorem, and define

$$f(a, b) = (a \vee j(a, b)) \wedge b,$$

where

$$j(a, b) = k(D_{a,b}, e_{a,b}, 0)$$
$$D_{a,b} = \{a,\ b,\ j(a', b') : \langle a', b' \rangle < \langle a, b \rangle\}$$

How to compute the various $\varphi_{e_{a,b}}(0)$:

$j(a,b) = k(D_{a,b}, e_{a,b}, 0)$, $D_{a,b} = \{a,\ b,\ j(a',b') : \langle a',b'\rangle < \langle a,b\rangle\}$ Sketch II

Recall: $f(a,b) = (a \vee j(a,b)) \wedge b$, $\quad j(a,b) = k(D_{a,b}, e_{a,b}, 0)$.

At step s we consider all pairs (a,b) with $\langle a,b\rangle \leq s$, for which $\varphi_{e_{a,b}}(0)$ is still undefined:

1. for each such pair (a,b) let $\langle a_m, b_m\rangle < \langle a,b\rangle$ (if it exists) have least Cantor number in the $=^2_L$-class of (a,b). If so, define $\varphi_{e_{a,b}}(0) = j(a_m, b_m)$. As k is a totalizer, this makes $j(a,b) =_L j(a_m,b_m)$ and thus $f(a_m,b_m) =_L f(a,b)$;
2. if after this, $\langle a,b\rangle \leq s$ is still a pair such that $\varphi_{e_{a,b}}(0)$ is undefined and $a \leq_L b$ at s then
   - if $f(a,b) =_L a$ (at s) then define $\varphi_{e_{a,b}}(0) = b$: this forces $j(a,b) =_L b$ and thus $a =_L b$;
   - if $f(a,b) =_L b$ (at s) then define $\varphi_{e_{a,b}}(0) = a$: this forces $j(a,b) =_L a$ and thus, again, $a =_L b$.

Verifications:

- f is well defined on $=^2_L$-classes by Clause 1: If $a =_L b$ then $f(a,b) =_L a$, and thus f is well defined in this case.
  Assume $a \neq_L b$: then as Clause 2 does not happen (since it would force $a =_L b$), we have that $f(a,b) =_L f(a_m, b_m)$ where (a_m, b_m) has least Cantor number in the $=^2_L$-class of (a,b) at stage s.

- If $a <_L b$ then $a <_L f(a,b) <_L b$: notice that we have $a \leq_L f(a,b) \leq_L b$ for free as $a \leq_L b$. As f is well defined on $=^2_L$-class, we may assume that $\langle a, b \rangle$ is least among the pairs in the $=^2_L$-class of (a,b). So we never use Clause 1 to define $f(a,b)$.
  If $a <_L f(a,b) <_L b$ does not hold then time will come when Clause 2 would force $a =_L b$, a contradiction. □

Definition A lattice of sentences is a positive lattice $L_{\mathcal{C},T}$, where T is a (classical or intuitionistic) formal system; the universe $\mathcal{C}$ is presented by a c.e. set of sentences identified with ω, with operations induced by the propositional connectives $\vee$ and $\wedge$; the pre-ordering relation $\leq_{L_{\mathcal{C},T}}$ is induced by $\rightarrow_T$:

$$\alpha \leq_{L_{\mathcal{C},T}} \beta \text{ if } T \vdash \alpha \rightarrow \beta,$$

$$\alpha <_{L_{\mathcal{C},T}} \beta \text{ if } T \vdash \alpha \rightarrow \beta \text{ but } T \nvdash \beta \rightarrow \alpha;$$

and its equality relation $=_{L_{\mathcal{C},T}}$ induced by $\leftrightarrow_T$:

$$\alpha =_{L_{\mathcal{C},T}} \beta \text{ if } T \vdash \alpha \leftrightarrow \beta.$$

Example (Motivating Example)

T is any classical consistent c.e. extension of Robinson's Q (or R), and $\mathcal{C} = \Sigma_n$-sentences, for some $n \geq 1$, or $\mathcal{C}$=all sentences.

Applications of effective inseparability to lattices of sentences, I

We shall consider positive bounded lattices of sentences $L_{\mathcal{C},T}$, where, via coding, $0_{L_{\mathcal{C},T}}$ consists of the sentences of $\mathcal{C}$ refuted by T, and $1_{L_{\mathcal{C},T30}}$ consists of the sentences of $\mathcal{C}$ proved by T.

By the Local Universality Theorem and the Uniform Density Theorem, one can show that $\leq_{L_{\mathcal{C},T}}$ is locally universal and uniformly dense by simply showing that the pair $(0_{L_{\mathcal{C},T}}, 1_{L_{\mathcal{C},T}})$ is e.i..

So it is possible to derive, using only computability-theoretic methods, results on density and uniform density relative to well known lattices of sentences.

See the paper by [Shavrukov and Visser 2014] for a thorough investigation of uniform density and lattices of sentences, and the relevance of this topic to proof theory and logic.

How to prove e.i.-ness

In most cases it is enough to show that for each disjoint pair (A_0, A_1) of c.e. sets, we can find "polynomials" s_0, s_1, t_0, t_1 such that, taking

$$\beta(x) := \exists y\, \exists \vec{y} \leq y\, (s_0(x, \vec{y}) = t_0(x, \vec{y}) \\ \wedge \forall \vec{z} \leq y)\, (s_1(x, \vec{z}) \neq t_1(x, \vec{z})))$$

we have $(A_0, A_1) \subseteq (X_0, X_1)$, where

$$X_0 = \{n : T \vdash \beta(\overline{n})\},$$
$$X_1 = \{n : T \vdash \neg\, \beta(\overline{n})\}.$$

Since $(X_0, X_1) \leq_m (1_{L_T}, 0_{L_T})$, it is clear that if (A_0, A_1) was chosen to be e.i. then the pair $(0_{L_T}, 1_{L_T})$ is e.i..

V. Yu. Shavrukov and A. Visser. Uniform density in Lindenbaum algebras. Notre Dame J. Form. Log., 55(4):569–582, 2014

D. Pianigiani and A. Sorbi. A note on uniform density in weak arithmetical theories. Arch. Math. Logic, 2021

Local universality and uniform density for lattices of sentences $L_{\mathcal{C},T}$, known cases I

Already known examples:

1 the lattice $L_{\mathcal{C},T}$ is known already to be locally universal and uniformly dense if $L_{\mathcal{C},T}$ is an e.i. Boolean algebra: for instance if T is a consistent c.e. extension of Q or R, and $\mathcal{C} = \Delta_n$, with $n \geq 2$, or $\mathcal{C}$=all sentences.

For local universality see [Montagna and S.]; for uniform density [Shavrukov and Visser].

2 $=_{L_{\mathcal{C},T}}$ is uniformly dense if $=_{L_{\mathcal{C},T}}$ is a precomplete ceer: for instance if T is a consistent c.e. extension of $I\Delta_0 + \exp$ and $\mathcal{C} = \Sigma_n$ with $n \geq 1$.

See [Shavrukov and Visser].

Precompleteness is a stronger notion than u.f.p.: an equivalence relation E is precomplete if there exists a computable function $f(e, x)$ such that

$$(\forall e, x)\,[\varphi_e(x)\downarrow\Rightarrow \varphi_e(x)\ E\ f(e, x)]\,.$$

If T is a consistent c.e. extension of $I\Delta_0 + \exp$ and $n \geq 1$ then $L_{\Sigma_n, T}$ is precomplete because T has a truth predicate for the Σ_n formulas.

New examples using effective inseparability of $(0_{L_\mathcal{L}}, 1_{L_\mathcal{L}})$

In addition to local universality also for the previous examples when not already known notice:

1 If T is any consistent c.e. extension of Q or R and $C = \Sigma_n$ with $n \geq 1$, then
$L_{\mathcal{C},T}$ is locally universal and uniformly dense.

This was open, since Shavrukov and Visser's proof for $I\Delta_0 + \exp$, based on prencompleteness, cannot be applied, because it is still an open problem whether equality of $L_{\Sigma_1,Q}$ is precomplete.

2 If $T$is any consistent c.e. extension of Buss's weak system of arithmetic S^1_2, then $L_{\exists\Sigma^b_1,T}$ is locally universal, and (solving a problem in [Shavrukov and Visser]) uniformly dense.

Local universality and uniform density, new cases II

How about lattices of sentences of the form $L_{\mathcal{C},T}$, where T is an intuitionistic consistent c.e. extension of iQ or iR?

Of course Heyting Arithmetic L_{HA} is locally universal and uniformly dense, being an e.i. lattice.

For classes of sentences, The problem here is that it does not make too much sense to talk about $\mathcal{C}$ being Σ_n, because this class is not closed, modulo provable equivalence, under connectives.

We must look at different hierarchies of formulas, for example at Burr's hierarchies, and choose among the classes $\mathcal{C}$ of sentences which are closed under connectives.

Of course,

For instance if $\mathcal{C} = \Phi_n$ with $n \geq 3$, or $\mathcal{C} = \Theta_n$ with $n \geq 2$ (these classes refer to Burr's hierarchies of formulas) then $L_{\mathcal{C},T}$ is locally universal and uniformly dense.

$\Phi_0 := \Delta_0$
$\Phi_1 := \Sigma_1$
$\Phi_2 := \Pi_2$
for $n \geq 2$, let Φ_{n+1} be inductively defined by

- $\Phi_n \subseteq \Phi_{n+1}$
- if $\varphi \in \Phi_n, \psi \in \Phi_{n+1}$ then $\varphi \to \psi \in \Phi_{n+1}$
- if $\varphi \in \Phi_{n+1}$ then $(\forall x)\varphi \in \Phi_{n+1}$
- if $\varphi, \psi \in \Phi_{n+1}$ then $\varphi \wedge \psi, \varphi \vee \psi \in \Phi_{n+1}$
- if $\varphi \in \Phi_{n-1}$ then $(\exists x)\varphi \in \Phi_{n+1}$.

$\Theta_0 = \Delta_0$
$\Theta_1 = \Sigma_1$
For $n \geq 1$

- $\Theta_n \subseteq \Theta_{n+1}$
- Θ_{n+1} is closed under $\wedge, \vee, \exists, \forall$
- if $\varphi \in \Theta_n$ and $\psi \in \Theta_{n+1}$ then $\varphi \to \psi \in \Theta_{n+1}$.

W. Burr. The intuitionistic arithmetical hierarchy. In Logic Colloquium '99, volume 17 of Lect. Notes Log., pages 51–59. Assoc. Symbol. Logic, Urbana, IL, 2004

We know that all e.i. Boolean algebras are computably isomorphic, but nothing like this holds of lattices.

We have that $L_{\Sigma_1,PA}$ is not computably isomorphic with $L_{\Sigma_n,PA}$, for any $n > 1$ (in fact there is no isomorphism at all). This is true because PA is Σ_1-valid, and then the top element of $L_{\Sigma_1,PA}$ is join-irreducible, whereas if $n > 1$ then the top element of $L_{\Sigma_n,PA}$ is join-reducible.

Problem [[Shavrukov-Visser]] Are all the e.i. lattices $L_{\Sigma_n,PA}$ computably isomorphic with each other for all $n > 1$?

Notice however:

Theorem [[Montagna and S.]] For $n > 1$ these lattices $L_{\Sigma_1,PA}$ are universal with respect to the class of of positive bounded distributive lattices.

Problem Can effective inseparability help to provide more embedding results of positive lattices (and not only positive preorderings) into e.i. lattices?

Thanks

Thanks to Andrew and the organizers for this delightful and precious meeting . . .

. . . and thanks, Ted, for everything!