The topological α-game metatheorem

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Thm: [Solecki 98 for Δ^0_2] [Zapletal 04 for Borel]
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Theorem: [Marks, Montalbán]
For every ordinal β and Borel function $F\colon \omega^\omega \to \mathcal{X}$,
the following are equivalent:

- F is not piecewise Baire-class-β.
- Every Baire-class-$(\beta+1)$ function continuously reduces to F.

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Theorem: [Marks, Montalbán – work in progress]

For every ordinal β and every Borel function F, TFAE:

- The pre-image under F of every $\mathbf{\Sigma}^0_{\beta+1}$ set is $\mathbf{\Sigma}^0_{\beta+1}$.
- F is piecewise continuous on Π^0_β domains,

Summary

1. Applications
2. The statement of the topological α-game metatheorem
3. An example: Wadge's theorem
4. The predecessor: the game metatheorem for structures
5. The general version: using α-topology
6. The key-ingredient in the proof: α-true stages

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Theorem: Let M, N be Σ^1_1 disjoint s.t. for no $\mathbf{\Sigma}^0_{\alpha+1}$ set C, $M \subseteq C \subseteq N^c$.
Then, for every $\Pi^0_{\alpha+1}$ set D, there is a continuous $F: (D, D^c) \to (M, N)$.

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Proof: We build a strategy for the engineer such that $P \circ \mathcal{T}^\alpha : (D, D^c) \to (M, N)$.
By metatheorem, the extender can then ensure that $F = P \circ \mathcal{T}^\alpha : \omega^\omega \to \mathcal{X}$ is continuous.

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Then, for every $\Pi^0_{\alpha+1}$ set D, there is a continuous $F: (D, D^c) \to (M, N)$.

Proof: We build a strategy for the engineer such that $P \circ \mathfrak{T}^\alpha : (D, D^c) \to (M, N)$.
By metatheorem, the extender can then ensure that $F = P \circ \mathfrak{T}^\alpha : \omega^\omega \to \mathcal{X}$ is continuous.

We want P such that $G(y) = 0^\omega \to P(y) \in M$ and $G(y) \neq 0^\omega \to P(y) \in N$ for $y = \mathfrak{T}^\alpha(x)$.

Let $H: \omega^\omega \to 2^\omega$ be a $\Sigma^0_{\alpha+1}$ function s.t. $(\forall x \in \omega^\omega)\ x \in D \iff H(x) = 0^\omega$.
Let $G: \omega^\omega \to 2^\omega$ be continuous such that $H = G \circ \mathfrak{T}^\alpha$. Extend to $G: \omega^{\leq\omega} \to 2^{\leq\omega}$.

We want, for $\tau \in \omega^{<\omega}$, $G(\tau) = 0...0 \to A_\tau \subseteq A_0$ and $G(\tau) \neq 0...0 \to A_\tau \subseteq N$.

Engineer's strategy: for $\tau \in \omega^{<\omega}$, define $A_\tau = \begin{cases} B_{\tau^-} \cap A_{\tau^-} & \text{if } G(\tau) = 0...0 \\ B_{\tau^-} \cap N & \text{if } G(\tau) \neq 0...0 \end{cases}$ $(\tau^- = \tau \restriction |\tau| - 1)$

$A_0 = M \setminus \bigcup\{B \in \Pi^0_\alpha(HYP) : B \subseteq N^c\}$. $A_0 \neq \emptyset$, A_0 is Σ^1_1, and $B_\tau \cap A_0 \neq \emptyset \to B_\tau \cap N \neq \emptyset$

Wedge's theorem from the $(\mathcal{X}, \mathbb{A}, \mathbb{A}_\alpha)$-game metatheorem

$\mathbb{A} = \Sigma^1_1$, $\mathbb{A}_\alpha = \Pi^0_\alpha(HYP)$, $\mathfrak{T}^{\alpha+1}: \omega^\omega \to \omega^\omega$ left-complete $\Sigma^0_{\alpha+1}$, $P: \omega^\omega \to \mathcal{X}$, $P(Y) = \bigcap_{\tau \subset Y} B_\tau$

The engineer: $\tau \mapsto A_\tau : \omega^{<\omega} \to \mathbb{A}$ such that $\tau' \subset \tau \implies B_{\tau'} \supseteq A_\tau$.

The extender: $\tau \mapsto B_\tau : \omega^{<\omega} \to \mathbb{A}_\alpha$ such that $\tau' \subset \tau \implies B_{\tau'} \supseteq B_\tau$ and $A_\tau \cap B_\tau \neq \emptyset$.

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(Also take steps on the Gandy-Harrington forcing to make the intersection work out.)

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extender		$\bar{b}_0$		$\bar{b}_1$		$\bar{b}_2$		$\cdots$
oracle		n_0		n_1		n_2		$\cdots$
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...and many more theorems proved using Ash-Knight's η-systems.

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Theorem: [Louveau][MM] For the collection, $\mathbb{A}$, of lightface Σ^1_1 subsets:

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Let $\mathcal{X}$ be the space of presentations of structures, $\{\mathcal{A}_i : i \in \omega\} \subseteq \mathcal{X}$, and $\mathbb{A}$ the collection of $\{\mathcal{B} \in \mathcal{X} : (\mathcal{A}_i, \bar{a}) \cong (\mathcal{B}, \bar{n})\}$ for $i \in \omega$ and $\bar{a} \in A_i^{<\omega}$.
Then $\mathbb{A}_a$ consist of $\{\mathcal{B} \in \mathcal{X} : (\mathcal{A}_i, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{n})\}$ for $i \in \omega$ and $\bar{a} \in A_i^{<\omega}$.

A borader setting for the $(\mathcal{X}, \mathbb{A}, \mathbb{A}_\alpha)$-game: the α-topology

For a Polish space $\mathcal{X}$ and $\mathbb{A} \subseteq \mathcal{P}(\mathcal{X})$, we define $\mathbb{A}_\xi \subseteq \mathcal{P}(\mathcal{X})$ for $\xi \leq \alpha$:

- Let $\mathbb{A}_0$ be the basic open sets in the topology of $\mathcal{X}$.
- Let $\mathbb{A}_\xi = \mathbb{A}_{<\xi} \cup \{\overline{B}^\xi : B \in \mathbb{A}\}$,

where $\mathbb{A}_{<\xi} = \bigcup_{\zeta<\xi} \mathbb{A}_\zeta$ and
$\overline{B}^\xi$ be the closure of B in the topology generated by $\mathbb{A}_{<\xi}$.

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Theorem: [Greenberg, Turetsky 22] [M. 14] [M, CST book II] Such $\mathcal{T}^\xi$ and $\leq_\xi$, for $\xi \leq \alpha$, exist.

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Theorem: [Marks, Montalbán – work in progress]
For every ordinal β and Borel function $F\colon \omega^\omega \to \mathcal{X}$,
the following are equivalent:

- The pre-image under F of every $\mathbf{\Sigma}^0_{\beta+1}$ set is $\mathbf{\Sigma}^0_{\beta+1}$.
- F is piecewise continuous on Π^0_β domains.

HAPPY BIRTHDAY TED!