An analytic equivalence relation with an unexpected property

Or: Kumabe-Slaman, Forcing, and Me

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Borel graphability

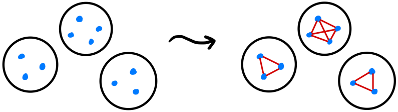
Definition (Arant). An equivalence relation E on a Polish space X is Borel graphable if there is a Borel graph G on X such that

$$x\,E\,y \iff \text{there is a path from } x \text{ to } y \text{ in } G.$$

Observation. Borel $\subseteq$ Borel graphable $\subseteq$ Analytic

Proof.

(1)



(2) Obvious.

(Easy) Question. Borel $\subsetneq$ Borel graphable $\subsetneq$ Analytic?

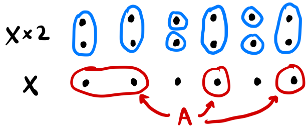
Answer. Yes.

Proposition. Borel graphable $\neq$ Analytic.

Proof. Let $A \subseteq X$ be an analytic set which is not Borel.

Define E on $X \times 2$ by

$$(x, i)\, E\, (y, j) \iff \begin{cases} x = y & \text{if } x \in A \\ x = y \text{ and } i = j & \text{if } x \notin A. \end{cases}.$$



Since all equivalence classes of E have size at most 2:

$$E \text{ Borel graphable} \iff E \text{ is Borel} \iff A \text{ is Borel.}$$

Comment. By Lusin-Novikov, if all equivalence classes of E are countable then E Borel graphable $\iff$ E is Borel

Proposition. Borel $\neq$ Borel graphable.

Proof. Let E be an analytic equivalence relation on X which is not Borel. Define E' on $X \times 2^\omega$ by

$$(x, a)\, E'\, (y, b) \iff x\, E\, y.$$

i.e. add a "dummy coordinate"

Define a graph G on $X \times 2^\omega$ by setting (x, a) and (y, b) adjacent if $a \oplus b$ computes a witness that $x\, E\, y$.

Comment. If E is Borel graphable then there is almost always a graphing given by setting x and y adjacent if $(x \oplus y \oplus a)^{(\alpha)}$ computes a witness to the E-equivalence of x and y for some fixed $a \in 2^\omega$ and $\alpha < \omega_1$.

A more interesting example

ω_1^{CK} = the least ordinal with no computable presentation
ω_1^x = the least ordinal with no presentation computable from x

Definition. F_{ω_1} is the equivalence relation on 2^ω defined by

$$x \, F_{\omega_1} \, y \iff \omega_1^x = \omega_1^y.$$

Some facts about F_{ω_1}.

(1) It is analytic (actually, Σ_1^1)

(2) Each equivalence class is Borel

(3) It has exactly $\aleph_1$ many equivalence classes

F_{ω_1} looks kind of like a counterexample to the topological Vaught's conjecture... but not that much.

Theorem (Marker). F_{ω_1} is not generated by any continuous Polish group action on 2^ω.

Theorem (Becker). Or even by any Borel Polish group action.

Definition. F_{ω_1} is the equivalence relation on 2^ω defined by

$$x \, F_{\omega_1} \, y \iff \omega_1^x = \omega_1^y.$$

Question. Is F_{ω_1} Borel graphable?

Answer. It depends!

Theorem (Arant-Kechris-L.). F_{ω_1} is Borel graphable if and only if there is a non-constructible real.

F_{ω_1} is not Borel graphable

Theorem (Arant). If all reals are constructible, F_{ω_1} is not Borel graphable.

Easier to prove:

Theorem (Arant). F_{ω_1} is not Δ^1_1-graphable.

Proof idea. Assume F_{ω_1} is Δ^1_1-graphable and show that the connected component of 0 is countable. Contradicts the fact that there are uncountably many x such that $\omega_1^x = \omega_1^0 = \omega_1^{CK}$.

Idea: Identify some countable set which contains 0 and has strong closure properties

Obvious choice: Δ^1_1.

Useful closure property: x is Δ^1_1 and $A \subseteq 2^\omega$ is $\Delta^1_1(x) \implies A$ is Δ^1_1.

Two key ingredients:

(1) Effective Perfect Set Theorem

(2) Friedman's Conjecture

Effective Perfect Set Theorem:

Theorem (Harrison). Every Σ^1_1 subset of 2^ω either contains a perfect set or every real in Δ^1_1.

Friedman's Conjecture (kind of):

Theorem (Martin/Friedman). If $A \subseteq 2^\omega$ is an uncountable Δ^1_1 set then for every $\alpha < \omega_1$, there is some $x \in A$ such that $\omega_1^x > \alpha$.

The point. If $A \subseteq 2^\omega$ is a Δ^1_1 set such that for every $x \in A$, $\omega_1^x = \omega_1^{CK}$ then every real in A is Δ^1_1.

Theorem (Arant). F_{ω_1} is not Δ^1_1-graphable.

Proof. Suppose G is a Δ^1_1-graphing of G. We will show that every real in the connected component of 0 is Δ^1_1.

Suppose y is in the connected component of 0 and $x_0, x_1, \ldots, x_n$ is a path from 0 to y. We will show by induction that each x_i is Δ^1_1.

Assume x_i is Δ^1_1. The set of neighbors of x_i,

$$A = \{z \mid x_i \text{ and } z \text{ are neighbors in } G\},$$

is $\Delta^1_1(x_i)$, hence Δ^1_1.

Since every $z \in A$ is in the same F_{ω_1}-equivalence class as x_i (and hence 0), every $z \in A$ has $\omega_1^z = \omega_1^{CK}$

So by the Effective Perfect Set Theorem/Friedman's Conjecture, every real in A is Δ^1_1. In particular, x_{i+1} is Δ^1_1.

Theorem (Arant). F_{ω_1} is not Δ^1_1-graphable.

Recall. Borel $= \Delta^1_1(a)$ for some $a \in 2^\omega$

Question. Why doesn't the proof relativize to give F_{ω_1} not Borel graphable?

Answer. The problem is in the application of Friedman's Conjecture

Friedman's Conjecture, relativized. If $A \subseteq 2^\omega$ is an uncountable $\Delta^1_1(a)$ set then for every $\alpha < \omega_1$, there is some $x \in A$ such that $\omega_1^{x \oplus a} > \alpha$.

But in general, ω_1^x can be much smaller than $\omega_1^{a \oplus x}$.

Solution. Use a special fact about reals in L.

Theorem (Guaspari/Kechris/Sacks). For every real $a \in L$, there is some real $b \in L$ such that:

(1) a is $\Delta^1_1(b)$

(2) and for all reals x, if $\omega_1^x \geq \omega_1^b$ then b is $\Delta^1_1(x)$ (and hence $\omega_1^x = \omega_1^{x \oplus b}$).

F_{ω_1} is Borel graphable

Theorem (Arant-Kechris-L.). If there is a non-constructible real then F_{ω_1} is Borel graphable.

Proof strategy. Let a be a non-constructible real.

Define a graph G by setting x and y adjacent if $x \oplus y \oplus a$ computes a witness that $\omega_1^x = \omega_1^y$.

Show that any two elements of 2^ω which are F_{ω_1}-equivalent are connected in G by a path of length 2.

Enough to show: Given $x, y \in 2^\omega$ such that $\omega_1^x = \omega_1^y$, we can find z such that $\omega_1^z = \omega_1^x$ and $z \oplus a$ computes witnesses that $\omega_1^x = \omega_1^z$ and that $\omega_1^y = \omega_1^z$.

Perfect tool to build z: Kumabe-Slaman forcing

Enough to show: Given $x, y \in 2^\omega$ such that $\omega_1^x = \omega_1^y$, we can find z such that $\omega_1^z = \omega_1^x$ and $z \oplus a$ computes witnesses that $\omega_1^x = \omega_1^z$ and that $\omega_1^y = \omega_1^z$.

Perfect tool to build z: Kumabe-Slaman forcing

Key property of Kumabe-Slaman forcing. Suppose we have

- M, a countable transitive model of ZFC
- a, a real not in M

Then there is some g Kumabe-Slaman generic over M such that $g \oplus a$ encodes essentially any information, including information about g itself

Theorem (L.-Siskind). For any countable transitive model M, any $x \in M$ and and any g Kumabe-Slaman generic over M, $\omega_1^{x \oplus g} = \omega_1^x$.

These two properties are enough to complete the proof under the assumption that $\omega_1^L = \omega_1$.

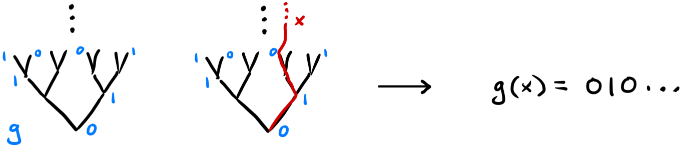
This assumption can be removed using an absoluteness argument (thanks to Gabe Goldberg)

Kumabe-Slaman forcing

End result of Kumabe-Slaman forcing. A partial labelling of $2^{<\omega}$.

i.e. a function $g\colon 2^{<\omega} \to \{0, 1, \perp\}$ ($\perp$ means “no label”)

Computational interpretation. Given a Kumabe-Slaman generic g and a real $x \in 2^{\omega}$, obtain a sequence in $2^{\leq\omega}$ byreading off the labels that g has placed along x. Denoted $g(x)$.



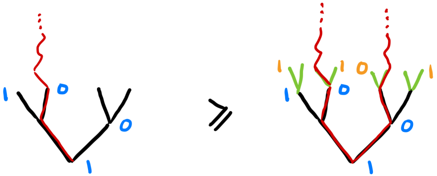
Intuitively: g encodes information on the path along x and adds labels on other paths through $2^{<\omega}$ in order to obfuscate where the information is hidden.

Conditions for Kumabe-Slaman forcing. A condition p for Kumabe-Slaman forcing consists of:

(1) A finite partial labelling $g_p\colon 2^{\leq n} \to \{0, 1, \perp\}$ for some n

(2) A finite set $X_p \subseteq 2^\omega$ of "forbidden paths."

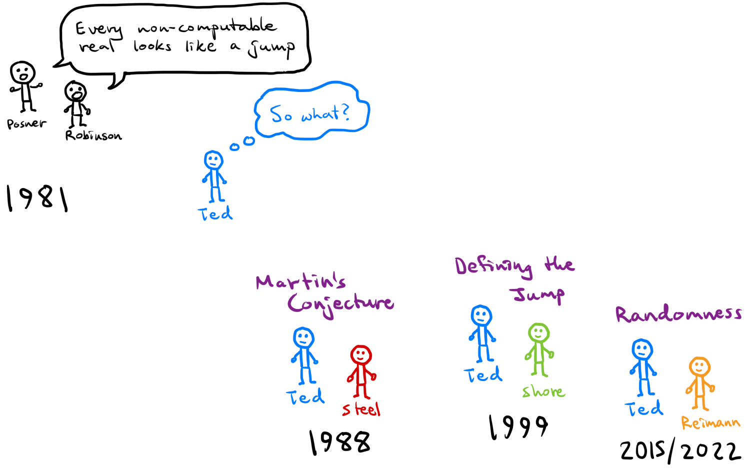
To extend p, we can add new forbidden paths and/or add new labels above n, as long as those new labels are not on any paths in X_p.



A brief history of Kumabe-Slaman forcing

Theorem (Posner-Robinson). For every noncomputable x, there is some g such that $x \oplus g \equiv_T g'$.

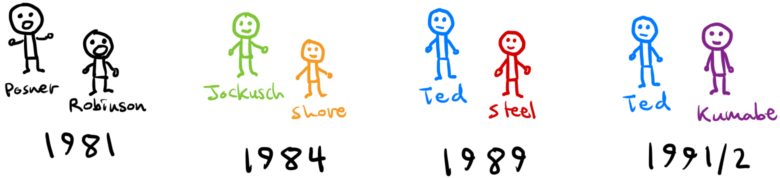
Intuitively: x "looks like the halting problem" relative to g.



Theorem (Posner-Robinson). For every noncomputable x, there is some g such that $x \oplus g \equiv_T g'$.

Natural question. Can this be extended? E.g. if x is not arithmetic, is there some g such that $x \oplus g \equiv_T g^{(\omega)}$?

A brief timeline:



Essentially resolved by Kumabe and Slaman in 1991 or 1992, via the invention of Kumabe-Slaman forcing.

Theorem (Posner-Robinson). For every noncomputable x, there is some g such that $x \oplus g \equiv_T g'$.

Key challenge in proving versions of the Posner-Robinson Theorem: You need to decide facts about g' without accidentally interfering with the coding of information into g.

Key fact about Kumabe-Slaman forcing. Suppose that we have:

- M, a countable, transitive model of ZFC
- $a \in 2^\omega$, a real not in M
- $p \in M$, a condition for Kumabe-Slaman forcing over M
- $D \in M$, a dense set for Kumabe-Slaman forcing over M.

Then we can find an extension q of p in M which meets D without adding any new labels along a

The proof consists of an ingenious use of compactness.

My involvement with Kumabe-Slaman forcing

Martin's Conjecture: A proposed classification of Turing invariant functions $2^\omega \to 2^\omega$.

Slaman and Steel: The Posner-Robinson Theorem can be used to prove special cases of Martin's Conjecture.

They used it to prove Martin's Conjecture for order-preserving functions which are below the hyperjump.

Generalizing this past the hyperjump requires two things:

(1) Proving further generalizations of the Posner-Robinson Theorem
(2) Showing that you can find witnesses to the Posner-Robinson Theorem which preserve ordinal valued functions on the Turing degrees.

It seems natural to use Kumabe-Slaman forcing for both.

Generalizing this past the hyperjump requires two things:

(1) Proving further generalizations of the Posner-Robinson Theorem
(2) Showing that you can find witnesses to the Posner-Robinson Theorem which preserve ordinal valued functions on the Turing degrees.

Day and Marks: Worked out item (1) using Kumabe-Slaman forcing

Siskind and Me: Currently working on item (2).

This was our motivation for:

Theorem (L.-Siskind). For any countable transitive model M, any $x \in M$ and and any g Kumabe-Slaman generic over M, $\omega_1^{x \oplus g} = \omega_1^x$.

It just happened to also be the exact theorem needed to analyze Borel graphability of F_{ω_1}!

Theorem (L.-Siskind). For any countable transitive model M, any $x \in M$ and and any g Kumabe-Slaman generic over M, $\omega_1^{x \oplus g} = \omega_1^x$.

The main idea in the proof is to prove:

Lemma. Suppose that M is an ω-model of ZFC and g is Kumabe-Slaman generic over V. Then g is also Kumabe-Slaman generic over M.
This is despite the fact that the forcing poset in M is different from the forcing poset in V.

The proof of the lemma follows the original analysis of Kumabe-Slaman forcing discovered by Kumabe and Slaman.