Degree Structures and Decidability

Steffen Lempp

University of Wisconsin-Madison

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Computability and Mathematical Definability

Celebrating the 70th Birthday of Ted Slaman

We will mostly consider subsets of ω and study their relative computational complexity.

Definition

- A *reducibility* is a transitive reflexive relation $\leq_r$ on $\mathcal{P}(\omega)$ (so that $A \leq_r B$ expresses that B "can compute" A).
- $A, B \subseteq \omega$ are *r-equivalent* (written $A \equiv_r B$) if $A \leq_r B$ and $B \leq_r A$. (A and B have "equal computational content".)
- The *r-degree* of A is $\deg_r(A) = \{B \mid A \equiv_r B\}$.
- The *global r-degree structure* is the partial order

  $$\mathcal{D}_r = (\mathcal{P}(\omega)/{\equiv_r}, \leq),$$

  where $\leq$ is induced by the *pre-partial order* $\leq_r$.
  We also consider *local r-degree structures*

  $$\mathcal{S}_r = (\mathcal{S}/{\equiv_r}, \leq)$$

  for a (usually countable) subfamily $\mathcal{S} \subset \mathcal{P}(\omega)$.

Many *reducibilities* have been considered in computability theory, e.g.:

- *Many-one reducibility:* $A \leq_m B$ if there is a computable function f such that for all x, $x \in A$ iff $f(x) \in B$.
- *Turing reducibility:* $A \leq_T B$ if there is a Turing functional Φ with $A = \Phi(B)$.
- *Enumeration reducibility:* $A \leq_e B$ if there is an enumeration operator Φ with $A = \Phi(B)$.
- *Ziegler reducibility* (or $*$-*reducibility*):
  $A \leq^* B$ if $A \leq_e B$ and $A^c \leq_e^1 B$ (defined on next slide).

All these lead to global (and many local) degree structures.

The local structures of particular interest arise when $\mathcal{S}$ is the family of c.e. sets, of Δ_2^0-sets, or of Σ_2^0-sets.

More general degree structures have also been defined on $\mathcal{P}(\mathcal{P}(\omega))$ (*Medvedev* and *Muchnik reducibility*) and on the set of partial multivalued functions from ω^ω to ω^ω (*Weihrauch reducibility*).

For completeness, here are precise definitions for some reducibilities:

Definitions

- $A \leq_T B$ if there is a Turing functional Φ with $A = \Phi(B)$, i.e., a c.e. set Φ of tuples (x, y, F, G) such that for all x and y, $A(x) = y$ iff there is $(x, y, F, G) \in \Phi$ with $F \subseteq B$ and $G \subseteq B^c$.
- $A \leq_e B$ if there is an enumeration operator Φ with $A = \Phi(B)$, i.e., a c.e. set Φ of pairs (x, F) such that for all x, $x \in A$ iff there is $(x, F) \in \Phi$ with $F \subseteq B$.
- $C \leq_e^1 B$ if there is a *1-enumeration operator* Ψ, i.e., a c.e. set Ψ of triples (x, F, G) such that for all x, $x \in C$ iff there is $(x, F, G) \in \Psi$ with $F \subseteq B$, $|G| \leq 1$ and $G \subseteq B^c$.
- $A \leq^* B$ if $A \leq_e B$ and $A^c \leq_e^1 B$.

Degree theory studies degree structures as algebraic objects, namely, as partial orders, sometimes in an expanded language.

For most "natural" degree structures $\mathcal{D}$, we have:

- $\mathcal{D}$ has a least element $\mathbf{0}_{\mathcal{D}}$.
- Local degree structures often have a greatest element, global degree structures usually do not.
- $\mathcal{D}$ is *locally countable*, i.e., any degree has at most countably many predecessors.
- $\mathcal{D}$ is an upper semilattice (but usually not a lattice), i.e., $\mathcal{D}$ has a join operation $\deg(A) \cup \deg(B) = \deg(A \oplus B)$, where $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.
- Global degree structures support a "jump" operation $\boldsymbol{a} \mapsto \boldsymbol{a}'$ such that $\boldsymbol{a} < \boldsymbol{a}'$, and $\boldsymbol{a} \leq \boldsymbol{b}$ implies $\boldsymbol{a}' \leq \boldsymbol{b}'$.

Most “natural” degree structures $\mathcal{D}$ are very complicated partial orders and usually follow this pattern:

- The first-order theory of the partial order $\mathcal{D}$ is undecidable. In fact, it is usually as complicated as second-order arithmetic (for global degree structures) or first-order arithmetic (for countable local degree structures).

Therefore, computability theorists often study “fragments” of the first-order theory, determined by a bound on the quantifier depth of the formulas:

- The $\exists$-theory of $\mathcal{D}$ is decidable (since all finite partial orders embed into $\mathcal{D}$).
- The $\forall\exists$-theory of $\mathcal{D}$ can “often” be shown to be decidable (more later).
- The $\exists\forall\exists$-theory of $\mathcal{D}$ can “usually” be shown to be undecidable (more later).

degree structure	complexity: 1st/2nd order arithmetic	∃-/∀∃-fragment decidable?	∃∀∃-fragment undecidable?
$\mathcal{D}_m$			
$\mathcal{D}_m(\leq \mathbf{0}'_m)$			
$\mathcal{D}_T$			
$\mathcal{D}_T(\leq \mathbf{0}'_T)$			
$\mathcal{D}_T(\text{c.e.})$			
$\mathcal{D}_e$			
$\mathcal{D}_e(\leq \mathbf{0}'_e)$			
$\mathcal{D}^*$			

For most degree structures, the undecidability of the first-order theory (in the language of partial order) was shown before the exact complexity of the full theory was determined.

In most cases, eventually the full theory turned out to have maximal complexity (that of second-order or first-order arithmetic) by coding a standard model of arithmetic (using parameters) into the degree structure.

These arguments tend to be quite complicated, using a fairly high quantifier complexity.

degree structure	complexity: 1st/2nd order arithmetic	∃-/∀∃-fragment decidable?	∃∀∃-fragment undecidable?
$\mathcal{D}_m$	2nd: Nerode, Shore 1980		
$\mathcal{D}_m(\leq \mathbf{0}'_m)$	1st: Nies 1994		
$\mathcal{D}_T$			
$\mathcal{D}_T(\leq \mathbf{0}'_T)$			
$\mathcal{D}_T(\mathrm{c.e.})$			
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degree structure	complexity: $1^{st}/2^{nd}$ order arithmetic	$\exists$-/$\forall\exists$-fragment decidable?	$\exists\forall\exists$-fragment undecidable?
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$\mathcal{D}_T(\mathrm{c.e.})$	1^{st}: Harrington, Slaman 1984		
$\mathcal{D}_e$	2^{nd}: Slaman, Woodin 1997		
$\mathcal{D}_e(\leq \mathbf{0}'_e)$			
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$\mathcal{D}_e(\leq \mathbf{0}'_e)$	1^{st}: Ganchev, M. Soskova 2012		
$\mathcal{D}^*$			

The undecidability of the ∃∀∃-theory can be usually shown via the

Nies Transfer Lemma 1996 (special case)

If a class $\mathcal{C}$ of finite relational structures is ∃-definable with parameters in $\mathcal{D}$, and the common ∀∃∀-theory of $\mathcal{C}$ is hereditarily undecidable, then the ∃∀∃-theory of $\mathcal{D}$ is undecidable.

The class $\mathcal{C}$ used in the results cited on the next slide is

- the class of all finite distributive lattices coded as initial segments for the *m*-degrees, the c.e. *m*-degrees, the Turing degrees, the Δ^0_2-Turing degrees, and the Ziegler degrees;

degree structure	complexity: 1st/2nd order arithmetic	∃-/∀∃-fragment decidable?	∃∀∃-fragment undecidable?
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$\mathcal{D}_e(\leq \mathbf{0}'_e)$	1st: Ganchev, M. Soskova 2012		
$\mathcal{D}^*$	undecidable (see right)		Jacobsen-Grocott, Lempp, I. Scott ta

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- the class of all finite bipartite graphs without equality with nonempty left and right domain in delicate coding arguments for the c.e. Turing degrees, the enumeration degrees, and the Σ_2^0-enumeration degrees.

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$\mathcal{D}_T(\text{c.e.})$	1st: Harrington, Slaman 1984		Lempp, Nies, Slaman 1998
$\mathcal{D}_e$	2nd: Slaman, Woodin 1997		Kent 2006
$\mathcal{D}_e(\leq \mathbf{0}'_e)$	1st: Ganchev, M. Soskova 2012		
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- the class of all finite distributive lattices coded as initial segments for the *m*-degrees, the c.e. *m*-degrees, the Turing degrees, the Δ^0_2-Turing degrees, and the Ziegler degrees; and
- the class of all finite bipartite graphs without equality with nonempty left and right domain in delicate coding arguments for the c.e. Turing degrees, the enumeration degrees, and the Σ^0_2-enumeration degrees.

(One can also code the class of all finite distrib. lattices as intervals in the enumeration degrees (Lempp, Slaman, M. Soskova 2021).)

Deciding the $\forall\exists$-theory of $\mathcal{D}$ amounts to giving a uniform decision procedure to the following

Algebraic Problem (for deciding the $\forall\exists$-theory of $\mathcal{D}$)

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_i \supseteq \mathcal{P}$ (for $i < n$), does every embedding of $\mathcal{P}$ into $\mathcal{D}$ extend to an embedding of $\mathcal{Q}_i$ into $\mathcal{D}$ for some $i < n$ (where i may depend on the embedding of $\mathcal{P}$)?

For the m-degrees and the c.e. m-degrees, one extends $\mathcal{P}$ minimally to a finite distributive lattice $\mathcal{L}$ and embeds it into $\mathcal{D}$ as an initial segment; now an embedding of $\mathcal{L}$ can be extended to an embedding of a finite partial order $\mathcal{Q}_i \supseteq \mathcal{L}$ iff no element of $\mathcal{Q}_i$ is below any element of $\mathcal{L}$, and $\mathcal{Q}_i$ respects joins in $\mathcal{L}$.

degree structure	complexity: 1st/2nd order arithmetic	∃-/∀∃-fragment decidable?	∃∀∃-fragment undecidable?
$\mathcal{D}_m$	2nd: Nerode, Shore 1980	∀∃: Dëgtev 1979	Nies 1996
$\mathcal{D}_m(\leq \mathbf{0}'_m)$	1st: Nies 1994		
$\mathcal{D}_T$	2nd: Simpson 1977		Lerman, Schmerl 1983
$\mathcal{D}_T(\leq \mathbf{0}'_T)$	1st: Shore 1981		
$\mathcal{D}_T(\text{c.e.})$	1st: Harrington, Slaman 1984		Lempp, Nies, Slaman 1998
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Deciding the ∀∃-theory of $\mathcal{D}$ amounts to giving a uniform decision procedure to the following

Algebraic Problem (for deciding the ∀∃-theory of $\mathcal{D}$)

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_i \supseteq \mathcal{P}$ (for $i < n$), does every embedding of $\mathcal{P}$ into $\mathcal{D}$ extend to an embedding of $\mathcal{Q}_i$ into $\mathcal{D}$ for some $i < n$ (where i may depend on the embedding of $\mathcal{P}$)?

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For the Turing degrees, one proceeds similarly but with a finite lattice $\mathcal{L}$ minimally extending $\mathcal{P}$.

degree structure	complexity: 1st/2nd order arithmetic	$\exists$-/$\forall\exists$-fragment decidable?	$\exists\forall\exists$-fragment undecidable?
$\mathcal{D}_m$	2nd: Nerode, Shore 1980	$\forall\exists$: Dëgtev 1979	Nies 1996
$\mathcal{D}_m(\leq \mathbf{0}'_m)$	1st: Nies 1994		
$\mathcal{D}_T$	2nd: Simpson 1977	$\forall\exists$: Lerman/ Shore 1978	Lerman, Schmerl 1983
$\mathcal{D}_T(\leq \mathbf{0}'_T)$	1st: Shore 1981		
$\mathcal{D}_T(\text{c.e.})$	1st: Harrington, Slaman 1984		Lempp, Nies, Slaman 1998
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For the m-degrees and the c.e. m-degrees, one extends $\mathcal{P}$ minimally to a finite distributive lattice $\mathcal{L}$ and embeds it into $\mathcal{D}$ as an initial segment; now an embedding of $\mathcal{L}$ can be extended to an embedding of a finite partial order $\mathcal{Q}_i \supseteq \mathcal{L}$ iff no element of $\mathcal{Q}_i$ is below any element of $\mathcal{L}$, and $\mathcal{Q}_i$ respects joins in $\mathcal{L}$.
For the Turing degrees, one proceeds similarly but with a finite lattice $\mathcal{L}$ minimally extending $\mathcal{P}$.
For the Δ_2^0-Turing degrees, embed $\mathcal{L}$ both as an initial segment; and also $\mathcal{L} - \{1\}$ as an initial segment, mapping 1 to $\mathbf{0}'_T$.

degree structure	complexity: $1^{st}/2^{nd}$ order arithmetic	∃-/∀∃-fragment decidable?	∃∀∃-fragment undecidable?
$\mathcal{D}_m$	2^{nd}: Nerode, Shore 1980	∀∃: Dëgtev 1979	Nies 1996
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$\mathcal{D}_e$	2^{nd}: Slaman, Woodin 1997		Kent 2006
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$\mathcal{D}^*$	undecidable (see right)		Jacobsen-Grocott, Lempp, I. Scott ta

degree structure	complexity: 1^{st}/2^{nd} order arithmetic	∃-/∀∃-fragment decidable?	∃∀∃-fragment undecidable?
$\mathcal{D}_m$	2^{nd}: Nerode, Shore 1980	∀∃: Dëgtev 1979	Nies 1996
$\mathcal{D}_m(\leq \mathbf{0}'_m)$	1^{st}: Nies 1994		
$\mathcal{D}_T$	2^{nd}: Simpson 1977	∀∃: Lerman/ Shore 1978	Lerman, Schmerl 1983
$\mathcal{D}_T(\leq \mathbf{0}'_T)$	1^{st}: Shore 1981	∀∃: Lerman, Shore 1988	
$\mathcal{D}_T(\text{c.e.})$	1^{st}: Harrington, Slaman 1984	∃: Sacks 1963	Lempp, Nies, Slaman 1998
$\mathcal{D}_e$	2^{nd}: Slaman, Woodin 1997	∃: Lagemann 1972	Kent 2006
$\mathcal{D}_e(\leq \mathbf{0}'_e)$	1^{st}: Ganchev, M. Soskova 2012		
$\mathcal{D}^*$	undecidable (see right)	∃: likely?	Jacobsen-Grocott, Lempp, I. Scott ta

Two major subproblems of the ∀∃-theory are the following:

Extension of Embeddings Problem

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q} \supseteq \mathcal{P}$, does every embedding of $\mathcal{P}$ into $\mathcal{D}$ extend to an embedding of $\mathcal{Q}$ into $\mathcal{D}$?

Lattice Embeddings Problem

Which finite lattices $\mathcal{L}$ can be embedded into $\mathcal{D}$ (preserving join and meet)?

The EE problem is decidable for the c.e. Turing degrees (Slaman, Soare 2001), for the enumeration degrees (Lempp, Slaman, M. Soskova 2021), and for the Σ^0_2-enumeration degrees (Lempp, Slaman, Sorbi 2005).
The LE problem remains open for the c.e. Turing degrees, but is decidable for the enumeration degrees and for the Σ^0_2-enumeration degrees (Lempp, Sorbi 2002: all finite lattices embed).

Given the difficulty of the overall problem of deciding the ∀∃-theory of the enumeration degrees and of the Σ^0_2-enumeration degrees, we are currently concentrating on the following subproblem of the Extension of Embeddings Problem for the Σ^0_2-enumeration degrees:

1-Point Extensions of Antichains

Decide, given a finite antichain $\mathcal{P} = \{a_0, \dots, a_n\}$ and 1-point extensions $\mathcal{Q}_S = \{a_0, \dots, a_n, x_S\}$ and $\mathcal{Q}^T = \{a_0, \dots, a_n, x^T\}$ for some *nonempty* subsets $S, T \subseteq \{0, \dots, n\}$ (where $x_S < a_i$ iff $i \in S$; and $x^T > a_i$ iff $i \in T$), whether any embedding of $\mathcal{P}$ can be extended to an embedding of $\mathcal{Q}_S$ for some such S or to an embedding of $\mathcal{Q}^T$ for some such T (not mapping the new element to $\mathbf{0}_e$ or $\mathbf{0}'_e$)?

(Note that it is always possible to extend an embedding of a finite antichain $\mathcal{P}$ to an embedding of the antichain $\mathcal{Q}_\emptyset = \mathcal{Q}^\emptyset$.)

The context for our subproblem is the two following earlier results:

Theorem (Ahmad 1989 (cf. Ahmad, Lachlan 1998))

1. There is an *Ahmad pair* of Σ^0_2-enumeration degrees $(\boldsymbol{a}, \boldsymbol{b})$, i.e., there are incomparable degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that any degree $\boldsymbol{v} < \boldsymbol{a}$ is $\leq \boldsymbol{b}$.
2. There is no *symmetric Ahmad pair* of Σ^0_2-enumeration degrees, i.e., there are no incomparable degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that any degree $\boldsymbol{v} < \boldsymbol{a}$ is $\leq \boldsymbol{b}$, and any degree $\boldsymbol{w} < \boldsymbol{b}$ is $\leq \boldsymbol{a}$.

These are examples of ∀∃-statements blocking $\mathcal{P} \subset \mathcal{Q}_0$ but not $\mathcal{P} \subset \mathcal{Q}_0, \mathcal{Q}_1$:

$\mathcal{P}$ $\mathcal{Q}_0$ $\mathcal{Q}_1$

We can handle the case of $\mathcal{Q}_S$:

Theorem in Progress (Goh, Lempp, Ng, M. Soskova)

Fix $n > 1$ and $\mathcal{S} \subseteq \mathcal{P}(\{0, \dots, n\}) - \{\emptyset\}$.
Let $S_0 = \{i \leq n \mid \{i\} \in \mathcal{S}\}$, and let $S_1 = \{0, \dots, n\} - S_0$.
Then some embedding of $\mathcal{P}$ into $\mathcal{D}_e(\leq \mathbf{0}'_e)$ cannot be extended to an embedding of $\mathcal{Q}_S$ for any $S \in \mathcal{S}$ iff

1. $S_0 = \emptyset$; or
2. $\bigcup \mathcal{S} \neq \{0, 1, \dots, n\}$; or
3. $S_1 \neq \emptyset$ and there is an *assignment* $\nu : S_0 \to \mathcal{P}(S_1) - \{\emptyset\}$, i.e., a function such that
    - for each $i \in S_0$, $\{i\} \cup \nu(i) \notin \mathcal{S}$, and
    - for each $F \subseteq S_0$ with $|F| > 1$, we have $\bigcap \{\nu(i) \mid i \in F\} \notin \mathcal{S}$.

The proof extends both results of Ahmad and combines them with minimal pair techniques.

As for $\mathcal{Q}^T$, we have to take into account the following

Theorem (Kalimullin, Lempp, Ng, Yamaleev 2022)

There is no cupping Ahmad pair, i.e., an Ahmad pair $(\boldsymbol{a}, \boldsymbol{b})$ with $\boldsymbol{a} \cup \boldsymbol{b} = \mathbf{0}'_e$.

We conjecture that this is the only additional obstruction when considering extensions by points above an antichain:

Conjecture

Fix $n > 1$ and $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(\{0, \dots, n\}) - \{\emptyset\}$.
Then some embedding of $\mathcal{P}$ into $\mathcal{D}_e(\leq \mathbf{0}'_e)$ cannot be extended to an embedding of $\mathcal{Q}_S$ for any $S \in \mathcal{S}$ or of $\mathcal{Q}^T$ for any $T \in \mathcal{T}$ iff

- $\mathcal{Q}_S$ satisfies the conditions of the Theorem in Progress, and
- any $T \in \mathcal{T}$ contains only one element, or contains two elements i, j with $j \in \nu(i)$ (from the Theorem in Progress).

Happy Belated Birthday,

and

Happy Retirement, Ted!