Ordinal Arithmetic without Σ^0_1 Induction

Marcia Groszek
Ben Logsdon
Justin Miller

Dartmouth College

Computability and Mathematical Definability
Celebrating the Seventieth Birthday of Theodore Slaman
University of California, Berkeley
October 11-13 2024

Context

Reverse Mathematics: Calibrate logical strength of theorems by set-theoretic existence axioms.

Use a first-order theory of second-order arithmetic.

RCA: P^- (finitary part of Peano Arithmetic), induction for all formulas, recursive (Δ^0_1) comprehension axiom.

RCA_0: Weaken induction to Σ^0_1 formulas.

RCA^*_0: Weaken induction to Δ^0_1 formulas; exponentiation is total.

Early reference for RCA_0^*

Factorization of polynomials and Σ_1^0 induction

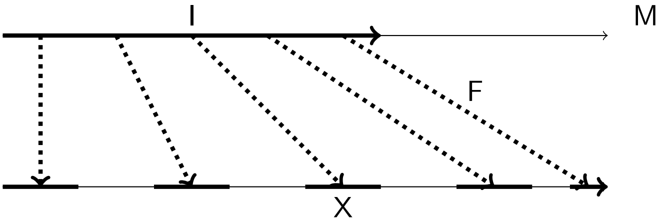
Stephen G. Simpson and Rick L. Smith

1986

Annals of Pure and Applied Logic

A model M of $\mathrm{RCA}_0^* + \neg I\Sigma_1^0$ has Σ_1^0-definable proper cuts.

I is Σ_1^0 definable but not an element of M.



F is increasing and cofinal with range X.

F and X are elements of M.

Ordinal arithmetic in RCA_0^*

A survey of the reverse mathematics of ordinal arithmetic

Jeffry L. Hirst

2005

Reverse Mathematics 2001 ed. S. G. Simpson

Overview

ATR_0: Ordinals behave well under addition, multiplication, exponentiation, ordering.

ACA_0: Ordering on ordinals is not total.

RCA_0: Exponentiation of ordinals is not total.

Overview

ATR_0: Ordinals behave well under addition, multiplication, exponentiation, ordering.

ACA_0: Ordering on ordinals is not total.

RCA_0: Exponentiation of ordinals is not total.

RCA_0^*: Multiplication of ordinals is not total.

Overview

Universal statements tend to persist in RCA_0^*

$$(\forall \alpha, \beta, \gamma)(\alpha^\beta \alpha^\gamma \cong \alpha^{\beta+\gamma})$$

unless they involve ordering

$$ATR_0 \iff (\forall \alpha)(\forall \beta)(\alpha \leq \beta \vee \beta \leq \alpha).$$

The ordinal ω is problematic.

Reversals

Suppose T is a theory extending RCA_0 and $RCA_0 \vdash T \leftrightarrow \varphi$.

To show $RCA_0^* \vdash T \leftrightarrow \varphi$:

Reversals

Suppose T is a theory extending RCA_0 and $RCA_0 \vdash T \leftrightarrow \varphi$.

To show $RCA_0^* \vdash T \leftrightarrow \varphi$:

When the given proof for the reversal requires $I\Sigma^0_1$,
show directly that $RCA_0^* + \neg I\Sigma^0_1 \vdash \neg\varphi$.

Reversals

Suppose T is a theory extending RCA_0 and $RCA_0 \vdash T \leftrightarrow \varphi$.

To show $RCA_0^* \vdash T \leftrightarrow \varphi$:

When the given proof for the reversal requires $I\Sigma_1^0$,
show directly that $RCA_0^* + \neg I\Sigma_1^0 \vdash \neg\varphi$.

Equivalence: Show $RCA_0^* \vdash I\Sigma_1^0 \leftrightarrow \varphi^*$ for a weak version φ^* of φ.

Reversals

Suppose T is a theory extending RCA_0 and $RCA_0 \vdash T \leftrightarrow \varphi$.

To show $RCA_0^* \vdash T \leftrightarrow \varphi$:

When the given proof for the reversal requires $I\Sigma_1^0$,
show directly that $RCA_0^* + \neg I\Sigma_1^0 \vdash \neg\varphi$.

Equivalence: Show $RCA_0^* \vdash I\Sigma_1^0 \leftrightarrow \varphi^*$ for a weak version φ^* of φ.

Local result: Characterize which numbers bound Σ_1^0-definable cuts using failures of φ (or of φ^*).

Example: Strong comparability of ordinals

Definition: $\alpha \leq_s \beta$ iff there is an order-preserving function from α onto an initial segment of β.

Theorem (RCA_0) (H. Friedman):

$ATR_0 \leftrightarrow$ For any ordinals α and β either $\alpha \leq_s \beta$ or $\beta \leq_s \alpha$.

Theorem (RCA_0^*):

(A.) If $I\Sigma_1^0$ does not hold, there are ordinals that are not strongly comparable.

Example: Strong comparability of ordinals

Definition: $\alpha \leq_s \beta$ iff there is an order-preserving function from α onto an initial segment of β.

Theorem (RCA_0) (H. Friedman):

$ATR_0 \leftrightarrow$ For any ordinals α and β either $\alpha \leq_s \beta$ or $\beta \leq_s \alpha$.

Theorem (RCA_0^*):

(A.) If $I\Sigma_1^0$ does not hold, there are ordinals that are not strongly comparable.

(B.) $I\Sigma_1^0$ holds iff any two ordinals, one of which is M-finite, are strongly comparable.

Example: Strong comparability of ordinals

Definition: $\alpha \leq_s \beta$ iff there is an order-preserving function from α onto an initial segment of β.

Theorem (RCA_0) (H. Friedman):

$ATR_0 \leftrightarrow$ For any ordinals α and β either $\alpha \leq_s \beta$ or $\beta \leq_s \alpha$.

Theorem (RCA_0^*):

(A.) If $I\Sigma_1^0$ does not hold, there are ordinals that are not strongly comparable.

(B.) $I\Sigma_1^0$ holds iff any two ordinals, one of which is M-finite, are strongly comparable.

(C.) A (nonstandard) number a bounds a Σ_1^0 cut iff there is an ordinal α that is not strongly comparable to a.

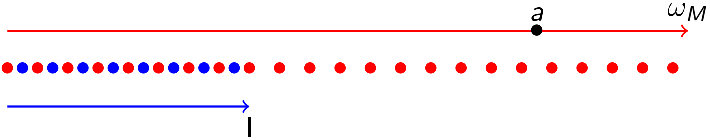
Example: Strong comparability of ordinals

a bounds a Σ^0_1 cut $\rightarrow$
there is an ordinal α that is not strongly comparable to a.

Choose I so $\frac{a}{2}$ bounds I (this is always possible);

$F : I \rightarrow M$ increasing, cofinal.

$\alpha = (\omega \times \{0\}) \cup graph(F)$, ordered lexicographically.



Key fact

Lemma (Chong and Mourad): If I is a Σ_1^0 cut in $M \models RCA_0^*$,
A is a Σ_1^0 subset of I, and $I - A$ is also Σ_1^0,
then there is an M-finite set X such that $A = X \cap I$.

Corollary: If I is closed under exponentiation,
M_I with universe I and second order part $\{X \cap I \mid X \text{ is } M\text{-finite}\}$
is a model of RCA_0^*.

Corollary: $M_I \models RCA_0$ iff I is a minimal Σ_1^0 cut.

One more reference

Weaker cousins of Ramsey's theorem over a weak base theory

Marta Fiori-Carones, Leszek Aleksander Kołodziejczyk, and Katarzyna W. Kowalik

2021

Annals of Pure and Applied Logic

The ordinal ω

The order type of M is ω_M.

If there is a minimal Σ_1^0-definable cut I_0, the order type of I_0 is ω_0.

Both are reasonable candidates for "ω."

Proposition (RCA_0^*):

ω_M^2 is an ordinal $\iff I\Sigma_1^0$.

ω_0^2 (if ω_0 exists) is always an ordinal.

There is an infinite ordinal α such that α^2 is also an ordinal

iff

there is a minimal Σ_1^0-definable cut.

Pushup and pullback

Suppose I_0 is a minimal Σ^0_1-definable cut.

M_{I_0} is denoted M_0

Let $F : I_0 \to M$ be increasing and cofinal with range X.

A structure S_0 on I_0 pushes up via F to a structure S on X in M.

A structure S on X in M pulls back via F to a structure S_0 in M_0.

These structures are isomorphic as second order structures in M and M_0 respectively.

Example: F takes ω_{M_0} in M_0 to ω_0 in M.

$\omega^2_{M_0}$ is an ordinal in M_0 because $M_0 \models RCA_0$,

Therefore ω^2_0 is an ordinal in M.

Example: Ordinals compared to ω

Theorem (RCA_0) (Friedman and Hirst, Hirst):

TFAE

(1.) ACA_0;

(2.) If α is an ordinal with $\omega \leq_w \alpha$ and $\alpha \leq_w \omega$ then $\omega \equiv_s \alpha$;

(3.) If α is an ordinal with $\omega \leq_w \alpha$ and $\alpha \not\leq_w \omega$ then $\omega <_w \alpha$.

Example: Ordinals compared to ω

Theorem (RCA_0) (Friedman and Hirst, Hirst):

TFAE

(1.) ACA_0;

(2.) If α is an ordinal with $\omega \leq_w \alpha$ and $\alpha \leq_w \omega$ then $\omega \equiv_s \alpha$;

(3.) If α is an ordinal with $\omega \leq_w \alpha$ and $\alpha \not\leq_w \omega$ then $\omega <_w \alpha$.

Proposition ($RCA_0^* + (\omega_0$ exists$)$):

TFAE

(1.) $M_0 \models ACA_0$;

(2.) If α is an ordinal with $\omega_0 \leq_w \alpha$ and $\alpha \leq_w \omega_0$ then $\omega_0 \equiv_s \alpha$;

(3.) If β is an ordinal with $\omega_0 \leq_w \beta$ and $\beta \not\leq_w \omega_0$ then $\omega_0 <_w \beta$.

Example: Ordinals compared to ω

(1.) $M_0 \models ACA_0$;

(3.) If β is an ordinal with $\omega_0 \leq_w \beta$ and $\beta \not\leq_w \omega_0$ then $\omega_0 <_w \beta$.

Suppose $M_0 \not\models ACA_0$.
Then (by Hirst) in M_0 there is a counterexample
$\omega_{M_0} \leq_w \alpha$ and $\alpha \not\leq_w \omega_{M_0}$ but $\omega_{M_0} \not<_w \alpha$.
That pushes up to a counterexample to (3) in M
$\omega_0 \leq_w \beta$ and $\beta \not\leq_w \omega_0$ but $\omega_0 \not<_w \beta$.

Suppose there is a counterexample to (3) in M
$\omega_0 \leq_w \beta$ and $\beta \not\leq_w \omega_0$ but $\omega_0 \not<_w \beta$.
Since $\omega_0 \not<_w \beta$, we must have $CARD(\beta) = \omega_0$.

Example: Ordinals compared to ω

(1.) $M_0 \models ACA_0$;

(3.) If β is an ordinal with $\omega_0 \leq_w \beta$ and $\beta \not\leq_w \omega_0$ then $\omega_0 <_w \beta$.

Suppose $M_0 \not\models ACA_0$.
Then (by Hirst) in M_0 there is a counterexample
$\omega_{M_0} \leq_w \alpha$ and $\alpha \not\leq_w \omega_{M_0}$ but $\omega_{M_0} \not<_w \alpha$.
That pushes up to a counterexample to (3) in M
$\omega_0 \leq_w \beta$ and $\beta \not\leq_w \omega_0$ but $\omega_0 \not<_w \beta$.

Suppose there is a counterexample to (3) in M
$\omega_0 \leq_w \beta$ and $\beta \not\leq_w \omega_0$ but $\omega_0 \not<_w \beta$.
Since $\omega_0 \not<_w \beta$, we must have $CARD(\beta) = \omega_0$.

I.e. there is $F : I_0 \to \beta$, so the counterexample pulls back to M_0, showing $M_0 \not\models ACA_0$.

Example: Ordinals compared to ω

Theorem (RCA_0) (Friedman and Hirst, Hirst):

TFAE

(1.) ACA_0;

(2.) If α is an ordinal with $\omega \leq_w \alpha$ and $\alpha \leq_w \omega$ then $\omega \equiv_s \alpha$;

(3.) If β is an ordinal with $\omega \leq_w \beta$ and $\beta \not\leq_w \omega$ then $\omega <_w \beta$.

Example: Ordinals compared to ω

Theorem (RCA_0) (Friedman and Hirst, Hirst):

TFAE

(1.) ACA_0;

(2.) If α is an ordinal with $\omega \leq_w \alpha$ and $\alpha \leq_w \omega$ then $\omega \equiv_s \alpha$;

(3.) If β is an ordinal with $\omega \leq_w \beta$ and $\beta \not\leq_w \omega$ then $\omega <_w \beta$.

Proposition (RCA_0^*):

TFAE

(1.) ACA_0;

(2.) If α is an ordinal with $\omega_M \leq_w \alpha$ and $\alpha \leq_w \omega_M$ then $\omega_M \equiv_s \alpha$.

Example: Ordinals compared to ω

Theorem (RCA_0) (Friedman and Hirst, Hirst):

TFAE

(1.) ACA_0;

(2.) If α is an ordinal with $\omega \leq_w \alpha$ and $\alpha \leq_w \omega$ then $\omega \equiv_s \alpha$;

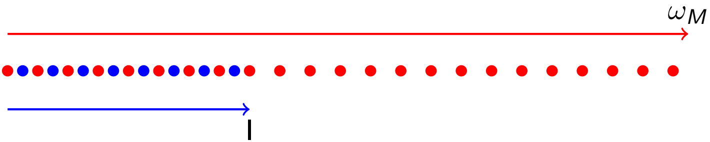
(3.) If β is an ordinal with $\omega \leq_w \beta$ and $\beta \not\leq_w \omega$ then $\omega <_w \beta$.

Proposition (RCA_0^*):

TFAE

(1.) ACA_0;

(2.) If α is an ordinal with $\omega_M \leq_w \alpha$ and $\alpha \leq_w \omega_M$ then $\omega_M \equiv_s \alpha$.



Example question

Suppose $RCA_0^* + \neg I\Sigma_1^0$.

Is there an ordinal β with $\omega_M \leq_w \beta$ and $\beta \not\leq_w \omega_M$ but $\omega_M \not<_w \beta$?

Example question

Suppose $RCA_0^* + \neg I\Sigma_1^0$.

Is there an ordinal β with $\omega_M \leq_w \beta$ and $\beta \not\leq_w \omega_M$ but $\omega_M \not<_w \beta$?

Over $RCA_0 + \neg ACA_0$, let X be Σ_1^0 and $X \notin M$.

Define β to contain a copy $x_0, x_1, \ldots$ of ω_M, with s-many elements between x_n and x_{n+1} if s is the least witness to $n \in X$.

Example question

Suppose $RCA_0^* + \neg I\Sigma_1^0$.

Is there an ordinal β with $\omega_M \leq_w \beta$ and $\beta \not\leq_w \omega_M$ but $\omega_M \not<_w \beta$?

Over $RCA_0 + \neg ACA_0$, let X be Σ_1^0 and $X \notin M$.

Define β to contain a copy $x_0, x_1, \ldots$ of ω_M, with s-many elements between x_n and x_{n+1} if s is the least witness to $n \in X$.

Then we cannot embed β in ω_M, because the image of x_{n+1} would give a bound on a witness to $n \in X$.

We cannot embed ω_M into an initial segment of β because initial segments of β are finite. This is by Σ_1^0 bounding, because initial segments of X are finite.

Example question

Suppose $RCA_0^* + \neg I\Sigma_1^0$.

Is there an ordinal β with $\omega_M \leq_w \beta$ and $\beta \not\leq_w \omega_M$ but $\omega_M \not<_w \beta$?

Over $RCA_0 + \neg ACA_0$, let X be Σ_1^0 and $X \notin M$.

Define β to contain a copy $x_0, x_1, \ldots$ of ω_M, with s-many elements between x_n and x_{n+1} if s is the least witness to $n \in X$.

Then we cannot embed β in ω_M, because the image of x_{n+1} would give a bound on a witness to $n \in X$.

We cannot embed ω_M into an initial segment of β because initial segments of β are finite. This is by Σ_1^0 bounding, because initial segments of X are finite.

That last fact requires $I\Sigma_1^0$.

Thank you!

Factorization of polynomials and Σ^0_1 induction
Stephen G. Simpson and Rick L. Smith
1986 *Annals of Pure and Applied Logic*

A survey of the reverse mathematics of ordinal arithmetic
Jeffry L. Hirst
2005 *Reverse Mathematics 2001* ed. S. G. Simpson

Weaker cousins of Ramsey's theorem over a weak base theory
Marta Fiori-Carones, Leszek Aleksander Kołodziejczyk, and Katarzyna W. Kowalik
2021 *Annals of Pure and Applied Logic*