The Minimal α-Degree Problem Revisited

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Recursion theory on admissible ordinals

A limit ordinal α is *admissible* if $(L_\alpha, \in) \models KP$.

- Given $A, B \subseteq \alpha$, we say that A is α-recursive in B ($A \leq_\alpha B$) if there is an algorithm for computing every α-finite subset of A and $\alpha \setminus A$ using α-finite information about B and $\alpha \setminus B$.
- $\leq_\alpha$ is a transitive relation.
- . $A \equiv_\alpha B$ means $A \leq_\alpha B$ and $B \leq_\alpha A$.
- $\equiv_\alpha$ decomposes subsets of α into equivalence classes called α-degrees.
- $\mathbf{0}$ is the α-degree of the α-recursive sets.
- $\mathbf{0}'$ is the α-degree of the halting set $\emptyset'$, etc.

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Brief overview

- (Sacks and Simpson, 1972) The Friedberg-Muchnik Theorem holds for all admissible α.
- (Lerman, 1974) There is a maximal α-r.e. set if and only if S_3-projectum$(\alpha) = \omega$.
- (Shore, 1976) The α-r.e. degrees are dense.
- (S Friedman, 1981) Assume $V = L$. If $\alpha = \aleph_{\omega_1}$, then the α-degrees $\geq \mathbf{0}'$ are well-ordered with successor generated via the jump operator. Every $\mathbf{a} \geq \mathbf{0}'$ is the α-degree of a master code.
- (Greenberg, Shore and Slaman, 2006) If $\alpha = \omega_1^{CK}$, then the ω-degree of the theory of α-r.e. degrees is that of $\mathcal{O}^{(\omega)}$.
- (Chong and Slaman, 2010) The theory of the α-degrees is undecidable for all α.

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Minimal α-degree

Definition

An α-degree $\mathbf{a} > \mathbf{0}$ is minimal if for all $\mathbf{b}$,

$$\mathbf{b} < \mathbf{a} \Rightarrow \mathbf{b} = \mathbf{0}.$$

- (Spector, 1956) There is a minimal ω-degree.
- (Sacks, 1963) There is a minimal ω-degree $< \mathbf{0}'$.
- (J Macintyre, 1973) If α is countable or a regular cardinal, then there is a minimal α-degree.
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The minimal α-degree problem

Prove that there is a minimal α-degree for every admissible α.

The Spector construction of a set of minimal ω-degree:

- Forcing with perfect trees to produce a generic G;
- Every oracle computation Φ is assigned with a recursive perfect tree T_Φ which is either “splitting” or “full”;
- For each Φ, G is a path on T_Φ.
- If Φ^G is total and T_Φ is a splitting tree, then $\Phi^G \equiv_T G$;
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The Spector technology

- The map
$$e \mapsto (\text{Index of})\, T_{\Phi_e}$$
can be made $\emptyset''$-recursive so as to obtain a set of minimal degree $<_T \mathbf{0}''$.
- By refining the construction with a $\emptyset$-recursive approximation, one can obtain a solution below $\mathbf{0}'$.
- This idea can be extended to handle Σ_2-admissible ordinals.

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Failure of the Spector idea

The approach fails for Σ_2-inadmissible α. As an example:

- Let $\alpha = \aleph_\omega^L$. For $n \in \omega$ define

$$\Phi_n^\sigma(x) = \begin{cases} \sigma(x) & \text{If } L_x \models \text{“There are less than } n \text{ cardinals”} \\ 1 & \text{Otherwise} \end{cases}$$

- For any G and n, Φ_n^G is α-recursive, and the Spector construction mandates T_{Φ_n} to be a full tree.
- Major obstruction:
  The set (of indices of) $\{\Phi_n : n \in \omega\}$ is α-finite but $\bigcap_{n\in\omega} T_{\Phi_n} = \{G\}$ is a single path and not an α-recursive perfect tree.
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Minimal α-degree for $\alpha = \aleph_{\omega_1}$ under $V = L$

Theorem ($V = L$)

If $\mathbf{a}$ *is a minimal* α*-degree, then* $\mathbf{a} < \mathbf{0}'$.

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Growth function of a set under $V = L$

Definition ($v = L$)

Let $A \subseteq \alpha = \aleph_{\omega_1}$. The growth function f_A of A is

$$f_A(x) = \text{the order of } A \restriction x \text{ in } L.$$

Definition

$A \subset \alpha$ is tame if there is a $B \leq_\alpha \emptyset'$ such that

$$\{\nu : \nu < \omega_1 \text{ and } f_A(\aleph_\nu) \leq f_B(\aleph_\nu)\}$$

is stationary in ω_1.

Lemma

If $A \subset \alpha$ is tame, then $A \leq_\alpha \emptyset'$.

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Minimal α-degree for $\alpha = \aleph_{\omega_1}$ under $V = L$

Lemma

For any $A \subset \alpha$ either deg(A) is not a minimal α-degree, or A is tame.

Below $\emptyset'$

A tree T is tagged with an α-recursive function $f : T \to \alpha$ if

- $\{f(\sigma) : \sigma \in T\}$ is unbounded in α;
- For all $\sigma \in T$, $f(\sigma) \leq |\sigma|$.

Definition

An α-recursive tree T tagged with f is quasi-splitting for Φ if

- For all $\sigma, \tau \in T$,

$$\sigma \upharpoonright f(\sigma) \neq \tau \upharpoonright f(\tau) \Rightarrow \exists x \leq \min\{f(\sigma), f(\tau)\}(\Phi^\sigma(x) \neq \Phi^\tau(x)).$$

Definition

An α-recursive tree T tagged with f is quasi-full for Φ if

- For all $\sigma, \tau \in T$,

$$\Phi^\sigma(x) = \Phi^\tau(x) \text{ for all } x \leq \min\{f(\sigma), f(\tau)\}.$$

Below $\emptyset'$

A tree T is tagged with an α-recursive function $f : T \to \alpha$ if

- $\{f(\sigma) : \sigma \in T\}$ is unbounded in α;
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Definition

An α-recursive tree T tagged with f is quasi-splitting for Φ if

- For all $\sigma, \tau \in T$,

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Below $\mathbf{0}'$

(Chong, 1979) The α-degree of $G \leq_\alpha \emptyset'$ is minimal if and only if

1. For each Φ, if Φ^G is total then there is an α-recursive tree T_Φ tagged with an f such that T_Φ is either quasi-splitting or quasi-full for Φ;
2. G is a path on T_Φ;
3. $$\{\min\{f(\sigma), f(\tau\} : \sigma, \tau \in T_\Phi \,\&\, \sigma \prec G, \tau \not\prec G\}$$

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When is the degree of A minimal?

Theorem ($V = L$)

Let $\alpha = \aleph_{\omega_1}$. Then $A \subset \alpha$ is of minimal α-degree if and only if

$$\{\nu : A \restriction \aleph_\nu \text{ is of minimal } \aleph_\nu\text{-degree }\}$$

is stationary in ω_1.

Corollary ($V = L$)

Let $\alpha = \aleph_{\omega_1}$. If $A \not\leq_\alpha \emptyset'$, then

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Countable *vs* uncountable cofinality

Corollary ($V = L$)

If there is a minimal $\aleph_{\omega_1}$-degree below $\mathbf{0}'$, then the set

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Conjecture:

Assume $V = L$. There is no minimal α-degree for $\alpha = \aleph_{\omega_1}$.

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