# Computable Structure Theory and Interactions
# Schedule and Abstracts

Version 1.0

15.7.2024–17.7.2024

Talks will be held in Freihaus Hörsaal 2, Wiedner Hauptstraße 8, 1040 Wien. To find the venue proceed to the yellow area (noticeable by the colored walls) of the building after entering, take the elevators or stairs to the second floor and proceed through the wooden door next to the elevators into a hallway leading to the auditorium.

| | Monday | Tuesday | Wednesday |
|---|---|---|---|
| 9:00-9:30 | Registration | | |
| 9:30-9:45 | Welcome | Coffee | Coffee |
| 9:45-10:30 | Gonzalez | Miller | Harizanov |
| 10:30-11:15 | Paolini | Block | Kalocinski |
| 11:15-11:30 | Coffee break | Coffee break | Coffee break |
| 11:30-12:15 | Scott | Villano | Ko |
| 12:15-13:30 | Lunch break | Lunch break | Lunch break |
| 13:30-14:15 | Łełyk | Vatev | Shafer |
| 14:15-18:00 | Problem session | Group work/Open | Group work/Open |

**Computable categoricity relative to a degree**

*Java Darleen Villano, University of Conneticut*, `javavill@uconn.edu`

A computable structure $\mathcal{A}$ is said to be computably categorical relative to a degree $\mathbf{d}$ if for all $\mathbf{d}$-computable copies $\mathcal{B}$ of $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$. In a 2021 result by Downey, Harrison-Trainor, and Melnikov, it was shown that there exists a computable graph $\mathcal{G}$ such that for an infinite increasing sequence of c.e. degrees $\mathbf{x}_0 <_T \mathbf{y}_0 <_T \mathbf{x}_1 <_T \mathbf{y}_1 \ldots$, $\mathcal{G}$ was computably categorical relative to each $\mathbf{x}_i$ but not computably categorical relative to each $\mathbf{y}_i$. In this talk, we will show how to extend this result for partial orders of c.e. degrees, and discuss some future directions of this project.

**Computable tree presentations of continuum-size structures**

*Russell Miller, Queens College - City University of New York*, `russell.miller@qc.cuny.edu`

Assorted attempts have been made to give a general framework for computability on uncountable structures. In this talk we will take one particular approach,exemplified by the usual presentation of the field of real numbers as limits of fast-converging Cauchy sequences, and extend it to a general definition of computability for structures of size continuum. (No assumption is made about the Continuum Hypothesis.) We will show how the automorphism group of a computable (countable) structure naturally fits this definition. In particular, recent work by the speaker on Galois groups of infinite algebraic field extensions (of countable fields) provides some quite congenial examples, as do the field

$\mathbb{R}$ itself and the ring $\mathbb{Z}_p$ of $p$-adic integers. The general principle is to name individual elements of the uncountable structure as paths through a computable subtree of $\omega^{<\omega}$.
Our definition is not without drawbacks, which we view as inherent in any attempt to present uncountable structures effectively via finite-time Turing computability. We will discuss ways to circumvent these difficulties. Time permitting, we will also present several natural questions that arise out of the new definition, some of which have well-established answers in certain cases.

**Approximately computable isomorphisms**
*Valentina Harizanov, George Washington University*, harizanv@gwu.edu
We study how different isomorphic copies of the structures relate through isomorphisms that are approximately computable, where approximate computability is based on the notion of asymptotic density. These isomorphisms include generically and coarsely computable isomorphisms, as well as their weaker variants and density preserving isomorphisms. Stronger and weaker variants of the notions result from the question whether there is a (total) isomorphism that is computable on a set of density one, or whether there is a partial computable isomorphism defined on a set of density one. Each of these notions provides an interesting insight. When computable isomorphic structures are not computably isomorphic, we investigate whether they are approximately computably isomorphic.
For example, we say that an isomorphism $F$ from a structure $A$ to a structure $B$ is *generically computable* if there is a substructure of $A$ with a computably enumerable universe $C$, with both $C$ and its image under $F$ being of asymptotic density one, and there is a partial computable function $\theta$ with domain $C$ such that $\theta$ is equal to $F$ on $C$. Considering computable Abelian $p$-groups, Goncharov and Smith independently characterized computably categorical ones. It follows from their characterization that there are computable Abelian $p$-groups that are countably infinite products of cyclic groups of order $p$ and $p^2$, which are not computably isomorphic. We give conditions under which two such groups are generically computably isomorphic.
This is joint work with Wesley Calvert and Doug Cenzer.

**Separating adaptability from disorderliness as randomness notions**
*Liling Ko, The Ohio State University*, ko.390@osu.edu
We separate an adaptive notion of randomness (having Kolmogorov-Loveland stochasticity zero) from a disorderly notion of randomness (having intrinsic density zero), and describe our combinatorial argument as an infinite variant of the Monty Hall game: There are infinitely many doors arranged in a line, and game show host $H$ hides a goat or car behind each one. Infinitely many doors hide cars. $H$ plays against countably many gamblers. After $H$ hides its cars, a gambler may select infinitely many doors to open, and wins if the proportion of cars in their selection is non-zero in the limit. As each gambler behaves like a computer program, the more "randomly" $H$ hides its cars, the more likely it can beat a gambler. We show that a host that beats gamblers who choose doors out of order may not necessarily beat gamblers who select doors adaptively, and vice-versa. This is joint work with Justin Miller.

**Computable Scott sentences and the weak Whitehead problem for finitely presented groups**
*Gianluca Paolini, University of Torino*, gianluca.paolini@unito.it
We prove that if $A$ is a computable Hopfian finitely presented structure, then $A$ has a computable $d$-$\Sigma_2$ Scott sentence if and only if the weak Whitehead problem for $A$ is decidable. We use this to infer that every hyperbolic group as well as any polycyclic-by-finite group has a computable $d$-$\Sigma_2$ Scott sentence, thus covering two main classes of finitely presented groups. Our proof also implies that every weakly Hopfian finitely presented group is strongly defined by its $\exists^+$-types, a question which arose in a different context.

**Complexity of cohesive powers**
*Paul Shafer, University of Leeds*, p.e.shafer@leeds.ac.uk

A *cohesive power* of a computable structure is an effective analog of an ultrapower where a cohesive set plays the role of an ultrafilter. We begin by surveying recent results on cohesive powers. We then investigate the complexity of cohesive powers. Every cohesive power of a computable structure via a $\Delta_2$ cohesive set has a $\Delta_3$ presentation. We show that the maximum complexity in this situation is achieved by exhibiting a computable graph $G$ where every presentation of every cohesive power of $G$ computes $0''$. This result is joint with David Gonzalez.

**Definability, Continuous embeddings and learning classes of algebraic structures**

*Stefan Vatev, University of Sofia*, `stefanv@fmi.uni-sofia.bg`

Definability in the logic $L_{\omega_1\omega}$ is closely related to Turing reducibility. For example, evaluating whether a computable infinitary $\Sigma^0_\alpha$ formula holds in a structure $\mathcal{A}$ is c.e. in the $\alpha$-th Turing jump of $\mathcal{A}$. It is also well known that continuous functions in the Cantor topology on $2^\omega$ are $X$-computable operators relative to some $X \in 2^\omega$, and thus Turing computable operators correspond to the effectively continuous functions in the Cantor topology.

On the other hand, Turing reducibility is not well suited to deal with partial objects, such as c.e. sets, partial functions, partial structures, etc. Enumeration reducibility was introduced by Friedberg and Rogers as a generalization of relative computability intended to deal with partial objects. It is natural to ask for a connection between definability and enumeration reducibility. This line of research was initiated by Soskov [5], where he considers only the positive part of the atomic diagram of a structure. Moreover, the continous functions in the Scott topology on $\mathcal{P}(\omega)$ are exactly the enumeration operators, relativized to some set $X$.

In this talk, I will explore some recent results [3] which give further insights into the dichotomy between Turing reducibility and enumeration reducibility. Most notably, there is a version of the Lopez-Escobar theorem that establishes a connection between the Borel hierarchy on the Scott topology on the space of countable structures and a hierarchy of positive $L_{\omega_1\omega}$-formulas. I will also discuss the effective counterparts of Borel embeddings, first studied in [1], given by Turing and enumeration operators, respectively. In the end, I will argue that this dichotomy can be transferred to algorithmic learning theory, where we have the natural counterparts of learning a class of structures from an informant, studied in [2], and learning a class of structures from a text, studied in [4].

*References:*

[1] J. Knight, S. Miller, and M. Vanden Boom: Turing Computable Embeddings, JSL, vol. 72 (2007), no. 3, pp. 901–918

[2] N. Bazhenov, E. Fokina, and L. San Mauro: Learning families of algebraic structures from informant, Information and Computation, vol. 275 (2020), article 104590.

[3] N. Bazhenov, E. Fokina, D. Rossegger, A. Soskova, and S. Vatev: A Lopez-Escobar Theorem for Continuous Domains, accepted in JSL (2024)

[4] N. Bazhenov, E. Fokina, D. Rossegger, A. Soskova, S. Vatev: Learning Families of Algebraic Structures from Text, Proceedings of Computability in Europe 2024

[5] I. Soskov: Degree Spectra and Co-spectra of Structures, Annual of Sofia University, vol. 96 (2004), pp. 45–68

**Scott ranks of models of arithmetic**

*Mateusz Łełyk, University of Warsaw*, `mlelyk@uw.edu.pl`

This a report on a joint work in progress with Patryk Szlufik. The talk concerns the recent results on the Scott ranks of models of Peano Arithmetic (PA) and its fragments. In "The structural complexity of models of arithmetic" Antonio Montalban and Dino Rossgger showed the rank of a model of PA is either 1 (in the case of the standard model) or is greater or equal to $\omega$ (and actually each of these possibilities is realised). Additionally, they investigated the relation between the model-theoretical properties of the given model and its Scott rank, showing for example that models of PA in which every element is definable have rank $\omega$ and homogeneous models have rank at most $\omega + 1$. They asked whether there are models of PA of rank $\omega$ which have undefinable elements.

We answer the question to the negative and outline the proof. We show how to modify the proof in order to characterize the models of PA of parametrized Scott rank $\omega$ and obtain some lower bounds on the Scott spectra of fragments of PA with the induction scheme restricted to $\Sigma_n$ formulae.

**Almost periodic Functions and the Scott Analysis of Linear Orderings**

*David Gonzalez, University of California, Berkeley*, `david_gonzalez@berkeley.edu`

The concept of Scott complexity was introduced by Alvir, Greenberg, Harrison-Trainor and Turetsky and gives a way of assigning countable structures to elements of the Borel hierarchy that correspond to their descriptive complexity. This concept refines the previous notions of Scott rank. In computable structure theory, Scott analysis refers to a wide variety of pursuits related to the concepts of Scott rank and Scott complexity. For example, it is typical to study the sorts of Scott ranks and Scott complexities that can appear in a given class of structures or the sorts of structures from a class that can have a given Scott rank or Scott complexity.

I will describe recent work that solves a number of open questions regarding the Scott analysis of linear orderings (and of structures in general). Central to this work is a new construction of a linear ordering given a so-called almost periodic function. We will discuss this construction and how to use the combinatorial structure of almost periodic functions to extract Scott analytic facts about their corresponding linear orderings.

This talk is based on joint work with Turbo Ho and Matthew Harrison-Trainor.

**Punctual presentability of trees**

*Dariusz Kalocinski, Polish Academy of Sciences*, `dariusz.kalocinski@gmail.com`

Punctual structure theory is a rapidly emerging subfield of computable structure theory which aims at understanding the primitive recursive content of algebraic structures. A countable structure is punctual if its domain is equal to the set of natural numbers and all its basic relations and functions are (uniformly) primitive recursive. One of the fundamental problems of this area is to understand whether computable members of a given class of structures admit a punctual presentation. We refer to such classes as punctually robust. In the talk I will discuss punctual robustness of trees. Crucially, trees can be represented as structures in various ways, for example as undirected graphs, partial orders or predecessor functions, among others. The main point of the talk will be to exhibit that the punctual robustness (or lack of it) of a class of trees depends on a mode of its structural representation. I will also discuss consequences of the presented results on punctual robustness of lattices and semilattices, undestood both in an order-theoretic and algebraic sense.

**Around existentially closed groups**

*Isabella Scott, University of Chicago*, `iscott@uchicago.edu`

Existentially closed groups were introduced in 1951 in analogue with algebraically closed fields. Since then, they have been further studied by Neumann, Macintyre, and Ziegler, who elucidated deep connections with model theory and computability theory. We will discuss how computability theory explains some of the structure of the class of e.c. groups and, if time allows, how these results translate (or not) to another class of structures.

**Elementarity of Subgroups and Complexity of Theories of Profinite Subgroups of $S_\omega$ via Tree Presentations**

*Jason Block, CUNY Graduate Center*, `jblock@gradcenter.cuny.edu`

Although $S_\omega$ (the group of all permutations of $\mathbb{N}$) is size continuum, both it and its closed subgroups can be presented as the set of paths through a countable tree. The subgroups of $S_\omega$ that can be presented this way with finite branching trees are exactly the profinite ones. We use these tree presentations to find upper bounds on the complexity of the existential theories of profinite subgroups of $S_\omega$, as well as to prove sharpness for these bounds. These complexity results enable us to distinguish a simple subclass of profinite groups, those with *orbit independence*, for which we find an upper bound on the complexity of the entire first order theory. Additionally, given a profinite subgroup $G$ of $S_\omega$ and a

Turing ideal $I$ we define $G_I$ to the be the set of elements in $G$ whose Turing degree lies in $I$. We examine to what degree and under what conditions $G_I$ will be an elementary subgroup of $G$.